

HARTMAN-WINTNER TYPE CRITERIA FOR HALF-LINEAR  
SECOND ORDER DIFFERENTIAL EQUATIONS

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*Abstract.* We establish Hartman-Wintner type criteria for the half-linear second order differential equation

$$(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-2}x, \quad p > 1,$$

where this equation is viewed as a perturbation of another equation of the same form.

*Keywords:* half-linear differential equation, Hartman-Wintner criterion, Riccati equation, principal solution

*MSC 2000:* 34C10

## 1. INTRODUCTION

Let us consider the half-linear second order differential equation

$$(1) \quad (r(t)\Phi(x'))' + c(t)\Phi(x) = 0,$$

where  $\Phi(x) := |x|^{p-1} \operatorname{sgn} x$ ,  $p > 1$ , and  $r, c$  are continuous functions,  $r(t) > 0$ .

This equation is a generalization of the second order Sturm-Liouville linear equation (with  $p = 2$  in (1))

$$(2) \quad (r(t)x')' + c(t)x = 0$$

and solutions of these two equations behave in many aspects very similarly. In particular, the oscillation theory extends almost verbatim from linear to half-linear equations and (1) can be classified as oscillatory or nonoscillatory according to whether

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any nontrivial solution of (1) has or does not have infinitely many zeros on any interval of the form  $[T, \infty)$ .

The classical Hartman-Wintner theorem for the nonoscillatory equation (2) (see, e.g. [7, p. 365]) relates the existence of the limit

$$(3) \quad \lim_{t \rightarrow \infty} \frac{\int^t r^{-1}(s) \left( \int^s c(\tau) d\tau \right) ds}{\int^t r^{-1}(s) ds}$$

to the convergence of the integral  $\int^\infty r^{-1}(t)w^2(t) dt$ , where  $w$  is any solution of the Riccati equation

$$w'(t) + c(t) + \frac{w^2(t)}{r(t)} = 0$$

associated with (2). As a consequence of this statement we have that (2) is oscillatory provided

$$-\infty < \liminf_{t \rightarrow \infty} \frac{\int^t r^{-1}(s) \left( \int^s c(\tau) d\tau \right) ds}{\int^t r^{-1}(s) ds} < \limsup_{t \rightarrow \infty} \frac{\int^t r^{-1}(s) \left( \int^s c(\tau) d\tau \right) ds}{\int^t r^{-1}(s) ds}$$

or if

$$\lim_{t \rightarrow \infty} \frac{\int^t r^{-1}(s) \left( \int^s c(\tau) d\tau \right) ds}{\int^t r^{-1}(s) ds} = \infty.$$

Consequently, an interesting problem is what is the oscillatory nature of (2) when the limit (3) exists and is finite. This problem was solved in [1], where it was shown that (2) (with  $r(t) \equiv 1$ ) is oscillatory provided

$$\limsup_{t \rightarrow \infty} \frac{t}{\ln t} \left( c(\infty) - \frac{1}{t} \int_1^t \int_1^s c(\tau) d\tau ds \right) > \frac{1}{4},$$

where

$$c(\infty) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_1^t \int_1^s c(\tau) d\tau ds.$$

Concerning the extension of these results to the half-linear case, the first step was made in Mirzov's paper [13] (see also [5], [14]), followed by [11], [12], where it was shown that these results naturally extend to (1). In particular, it was shown that (1) with  $r \equiv 1$  is oscillatory provided

$$\lim_{t \rightarrow \infty} c_p(t) = \infty \quad \text{or} \quad -\infty < \liminf_{t \rightarrow \infty} c_p(t) < \limsup_{t \rightarrow \infty} c_p(t),$$

where

$$(4) \quad c_p(t) = \frac{p-1}{t^{p-1}} \int_1^t s^{p-2} \int_1^s c(\tau) d\tau ds.$$

Moreover, if  $\lim_{t \rightarrow \infty} c_p(t) = c_p(\infty)$  exists and is finite and

$$\limsup_{t \rightarrow \infty} \frac{t^{p-1}}{\ln t} (c_p(\infty) - c_p(t)) > \left(\frac{p-1}{p}\right)^p,$$

then equation (1) is also oscillatory.

In the all above mentioned criteria, equation (1) is regarded as a perturbation of the one-term differential equation

$$(r(t)\Phi(x'))' = 0.$$

In this paper we consider equation (1) as a perturbation of a general (nonoscillatory) two-term equation

$$(5) \quad (r(t)\Phi(x'))' + \tilde{c}(t)\Phi(x) = 0,$$

i.e., (1) can be seen in the form

$$(r(t)\Phi(x'))' + \tilde{c}(t)\Phi(x) + (c(t) - \tilde{c}(t))\Phi(x) = 0.$$

We will investigate oscillatory properties of (1) depending on the asymptotic behaviour of the function

$$L(t) = \frac{\int_T^t H^{-1}(s) \left( \int_T^s (c(\tau) - \tilde{c}(\tau)) h^p(\tau) d\tau \right) ds}{\int_T^t H^{-1}(s) ds},$$

where  $H(t) = r(t)h^2(t)|h'(t)|^{p-2}$  and  $h(t)$  is the so-called principal solution of the nonoscillatory equation (5). By easy computation one can find that  $L(t)$  is a generalization of (3) and reduces to this function for  $p = 2, r(t) \equiv 1$  and  $\tilde{c} \equiv 0$  (it is well known that  $h = 1$  in this case, see [5, p. 146]).

## 2. PRELIMINARIES

Let  $x$  be a solution of (1). Then the function  $w = r\Phi(x'/x)$  solves the Riccati equation

$$(6) \quad w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0,$$

where  $q$  is the conjugate number of  $p$ , i.e.  $1/p + 1/q = 1$ , and it is well known (see [2, p. 171]) that equation (1) is nonoscillatory if and only if there exists a solution of (6) on some interval of the form  $[T, \infty)$ .

Now we recall the half-linear version of the so-called Picone's identity (see [10] or [2, p. 172]) which, in a modified form needed in our paper, reads as follows. Let  $w$  be a solution of (6). Then for any  $x \in C^1$

$$(7) \quad r(t)|x'|^p - c(t)|x|^p = (w(t)|x|^p)' + pr^{1-q}(t)P(r^{q-1}(t)x', \Phi(x)w(t)),$$

where

$$(8) \quad P(u, v) := \frac{|u|^p}{p} - uv + \frac{|v|^q}{q} \geq 0$$

with the equality  $P(u, v) = 0$  if and only if  $v = \Phi(u)$ .

Concerning the function  $P$ , we will need its quadratic estimates which are given in the next statement whose proof can be found e.g. in [6].

**Lemma 1.** *The function  $P(u, v)$  defined in (8) satisfies the inequalities*

$$\begin{aligned} P(u, v) &\geq \frac{1}{2}|u|^{2-p}(v - \Phi(u))^2 \quad \text{for } p \leq 2, \\ P(u, v) &\leq \frac{1}{2}|u|^{2-p}(v - \Phi(u))^2 \quad \text{for } p \geq 2, u \neq 0. \end{aligned}$$

Furthermore, let  $T > 0$  be arbitrary. There exists a constant  $K = K(T) > 0$  such that

$$\begin{aligned} P(u, v) &\geq K|u|^{2-p}(v - \Phi(u))^2 \quad \text{for } p \geq 2, \\ P(u, v) &\leq K|u|^{2-p}(v - \Phi(u))^2 \quad \text{for } p \leq 2 \end{aligned}$$

and every  $u, v \in \mathbb{R}$  satisfying  $|v/\Phi(u)| \leq T$ .

Now we derive the so-called modified Riccati equation. Let  $x \in C^1$  be any function and  $w$  a solution of the Riccati equation (6). Then from Picone's identity (7) we have

$$(9) \quad (w|x|^p)' = r|x'|^p - c|x|^p - pr^{1-q}|x|^pP(\Phi^{-1}(w_x), w),$$

where  $w_x = r\Phi(x'/x)$  and  $\Phi^{-1}$  is the inverse function of  $\Phi$ . At the same time, let  $h$  be a (positive) solution of (5) and  $w_h = r\Phi(h'/h)$  the solution of the Riccati equation associated with (5). Then

$$(10) \quad (w_h|x|^p)' = r|x'|^p - \tilde{c}|x|^p - pr^{1-q}|x|^pP(\Phi^{-1}(w_x), w_h).$$

Substituting  $x = h$  into (9), (10) and subtracting these equalities we get the equation (in view of the identity  $P(\Phi^{-1}(w_h), w_h) = 0$ )

$$(11) \quad ((w - w_h)h^p)' + (c - \tilde{c})h^p + pr^{1-q}h^pP(\Phi^{-1}(w_h), w) = 0.$$

Observe that if  $\tilde{c}(t) \equiv 0$  and  $h(t) \equiv 1$ , then (11) reduces to (6) and this is the reason why we call this equation the *modified Riccati equation*.

Further, let us recall the concept of the principal solution of the nonoscillatory equation (1) as introduced by Mirzov in [15] and later independently by Elbert and Kusano in [8]. If (1) is nonoscillatory, as mentioned at the beginning of this section, there exists a solution  $w$  of the Riccati equation (6) which is defined on some interval  $[T, \infty)$ . It can be shown that among all solutions of (6) there exists the *minimal* one  $\tilde{w}$  (sometimes called the *distinguished* solution), minimal in the sense that any other solution of (6) satisfies the inequality  $w(t) > \tilde{w}(t)$  for large  $t$ . Then the principal solution of (1) is given by the formula

$$\tilde{x} = K \exp \left\{ \int^t r^{1-q}(s) \Phi^{-1}(\tilde{w}(s)) ds \right\},$$

where  $K$  is a real nonzero constant, i.e., the principal solution  $\tilde{x}$  of (1) is a solution which “produces” the minimal solution  $\tilde{w} = r\Phi(\tilde{x}'/\tilde{x})$  of (6).

Finally, we present an important subsidiary statement, whose proof can be found in [3] or [4].

**Lemma 2.** *Let  $\int^\infty r^{1-q}(t) dt = \infty$ . Suppose that equation (5) is nonoscillatory and possesses a positive principal solution  $h$  such that there exists a finite limit*

$$(12) \quad \lim_{t \rightarrow \infty} r(t)h(t)\Phi(h'(t)) =: L > 0$$

and

$$(13) \quad \int^\infty \frac{dt}{r(t)h^2(t)(h'(t))^{p-2}} = \infty.$$

Further suppose that  $0 \leq \int_t^\infty c(s) ds < \infty$  for large  $t$ , (1) is nonoscillatory and

$$(14) \quad 0 \leq \int_t^\infty (c(s) - \tilde{c}(s))h^p(s) ds < \infty.$$

Then for any solution  $w$  of the Riccati equation (6) corresponding to (1) we have

$$\int^\infty r^{1-q}(t)h^p(t)P(\Phi^{-1}(w_h), w) dt < \infty \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{w(t)}{w_h(t)} = 1,$$

where  $w_h = r\Phi(h')/\Phi(h)$  is the solution of the Riccati equation corresponding to (5).

### 3. HARTMAN-WINTNER TYPE THEOREM

First we introduce the Hartman-Wintner type theorem, which is a completion of results published in [16]. The idea of our proof is similar to that used in [16], but for the sake of completeness and further references we include the proof.

**Theorem 1.** *Suppose that equations (1) and (5) are nonoscillatory and let  $h$  be a solution of (5) such that  $h'(t) \neq 0$  for large  $t$  and*

$$(15) \quad \int^{\infty} H^{-1}(t) dt = \infty, \quad H(t) := r(t)h^2(t)|h'(t)|^{p-2}.$$

*Let  $w$  be a solution of the Riccati equation (6) corresponding to (1) and  $w_h = r\Phi(h')/\Phi(h)$  a solution of the Riccati equation corresponding to (5) such that*

$$(16) \quad \limsup_{t \rightarrow \infty} \left| \frac{w(t)}{w_h(t)} \right| < \infty.$$

*Then for  $u(t) = h^p(t)(w(t) - w_h(t))$  and  $T$  sufficiently large the following statements are equivalent.*

(I) *The inequality*

$$(17) \quad \int_T^{\infty} \frac{u^2(t)}{H(t)} dt < \infty$$

*holds.*

(II) *There exists a finite limit*

$$(18) \quad \lim_{t \rightarrow \infty} \frac{\int_T^t H^{-1}(s) \int_T^s (c(\tau) - \tilde{c}(\tau))h^p(\tau) d\tau ds}{\int_T^t H^{-1}(s) ds}.$$

(III) *For the lower limit we have*

$$(19) \quad \liminf_{t \rightarrow \infty} \frac{\int_T^t H^{-1}(s) \int_T^s (c(\tau) - \tilde{c}(\tau))h^p(\tau) d\tau ds}{\int_T^t H^{-1}(s) ds} > -\infty.$$

**Proof.** (I  $\Rightarrow$  II): We can write (11) in the form

$$u'(t) + (c(t) - \tilde{c}(t))h^p(t) + pr^{1-q}(t)h^p(t)P(\Phi^{-1}(w_h), w) = 0.$$

Integrating from  $T$  to  $t$  we get

$$u(t) = u(T) - \int_T^t (c(s) - \tilde{c}(s))h^p(s) ds - p \int_T^t r^{1-q}(s)h^p(s)P(\Phi^{-1}(w_h), w) ds$$

and multiplying by  $H^{-1}$  and applying the same integration we obtain

$$\begin{aligned} \int_T^t H^{-1}(s)u(s) ds &= u(T) \int_T^t H^{-1}(s) ds - \int_T^t H^{-1}(s) \left( \int_T^s (c(\tau) - \tilde{c}(\tau))h^p(\tau) d\tau \right) ds \\ &\quad - p \int_T^t H^{-1}(s) \left( \int_T^s r^{1-q}(\tau)h^p(\tau)P(\Phi^{-1}(w_h), w) d\tau \right) ds \end{aligned}$$

and hence

$$\begin{aligned} \frac{\int_T^t H^{-1}(s)u(s) ds}{\int_T^t H^{-1}(s) ds} &= u(T) - \frac{\int_T^t H^{-1}(s) \left( \int_T^s (c(\tau) - \tilde{c}(\tau))h^p(\tau) d\tau \right) ds}{\int_T^t H^{-1}(s) ds} \\ &\quad - p \frac{\int_T^t H^{-1}(s) \left( \int_T^s r^{1-q}(\tau)h^p(\tau)P(\Phi^{-1}(w_h), w) d\tau \right) ds}{\int_T^t H^{-1}(s) ds}. \end{aligned}$$

Using the Cauchy-Schwartz inequality (suppressing the argument  $s$  in the integrated functions) we arrive at

$$0 \leq \frac{|\int_T^t H^{-1}u ds|}{\int_T^t H^{-1} ds} \leq \frac{[\int_T^t H^{-1} ds]^{\frac{1}{2}} [\int_T^t H^{-1}u^2 ds]^{\frac{1}{2}}}{\int_T^t H^{-1} ds} = \left( \frac{\int_T^t H^{-1}u^2 ds}{\int_T^t H^{-1} ds} \right)^{\frac{1}{2}} \rightarrow 0, \quad t \rightarrow \infty.$$

From Lemma 1 we know that provided (16) holds, there exist constants  $K_1, K_2$  such that

$$(20) \quad K_1 \frac{u^2}{H} \leq r^{1-q}h^p P(\Phi^{-1}(w_h), w) \leq K_2 \frac{u^2}{H}.$$

As  $\int_T^\infty H^{-1}u^2 dt < \infty$ , the integral  $\int_T^\infty r^{1-q}h^p P(\Phi^{-1}(w_h), w) dt$  converges too and by L'Hospital's rule we have

$$\lim_{t \rightarrow \infty} p \frac{\int_T^t H^{-1}(s) \left( \int_T^s r^{1-q}(\tau)h^p(\tau)P(\Phi^{-1}(w_h), w) d\tau \right) ds}{\int_T^t H^{-1}(s) ds} < \infty.$$

Therefore,

$$\begin{aligned} (21) \quad \lim_{t \rightarrow \infty} \frac{\int_T^t H^{-1}(s) \left( \int_T^s (c(\tau) - \tilde{c}(\tau))h^p(\tau) d\tau \right) ds}{\int_T^t H^{-1}(s) ds} \\ &= u(T) - \lim_{t \rightarrow \infty} p \frac{\int_T^t H^{-1}(s) \left( \int_T^s r^{1-q}(\tau)h^p(\tau)P(\Phi^{-1}(w_h), w) d\tau \right) ds}{\int_T^t H^{-1}(s) ds} \\ &= u(T) - p \int_T^\infty r^{1-q}(t)h^p(t)P(\Phi^{-1}(w_h), w) dt < \infty. \end{aligned}$$

(II  $\Rightarrow$  III): This implication is trivial.

(III  $\Rightarrow$  I): From the first part of this proof we have

$$\begin{aligned} \frac{\int_T^t H^{-1}(s)u(s) \, ds}{\int_T^t H^{-1}(s) \, ds} &= u(T) - \frac{\int_T^t H^{-1}(s) \left( \int_T^s (c(\tau) - \tilde{c}(\tau))h^p(\tau) \, d\tau \right) \, ds}{\int_T^t H^{-1}(s) \, ds} \\ &\quad - p \frac{\int_T^t H^{-1}(s) \left( \int_T^s r^{1-q}(\tau)h^p(\tau)P(\Phi^{-1}(w_h), w) \, d\tau \right) \, ds}{\int_T^t H^{-1}(s) \, ds}. \end{aligned}$$

The Cauchy-Schwartz inequality together with (19) and (20) implies that there exists a constant  $M \in \mathbb{R}$  such that

$$-\left( \frac{\int_T^t H^{-1}(s)u^2(s) \, ds}{\int_T^t H^{-1}(s) \, ds} \right)^{\frac{1}{2}} \leq M - pK_1 \frac{\int_T^t H^{-1}(s) \left( \int_T^s H^{-1}(\tau)u^2(\tau) \, d\tau \right) \, ds}{\int_T^t H^{-1}(s) \, ds}.$$

Suppose, by contradiction, that  $\int^\infty H^{-1}(t)u^2(t) \, dt = \infty$ . Then by L'Hospital's rule

$$\lim_{t \rightarrow \infty} \frac{\int_T^t H^{-1}(s) \left( \int_T^s H^{-1}(\tau)u^2(\tau) \, d\tau \right) \, ds}{\int_T^t H^{-1}(s) \, ds} = \infty$$

and

$$\begin{aligned} pK_1 \frac{\int_T^t H^{-1}(s) \left( \int_T^s H^{-1}(\tau)u^2(\tau) \, d\tau \right) \, ds}{\int_T^t H^{-1}(s) \, ds} - M \\ \geq \frac{1}{2} pK_1 \frac{\int_T^t H^{-1}(s) \left( \int_T^s H^{-1}(\tau)u^2(\tau) \, d\tau \right) \, ds}{\int_T^t H^{-1}(s) \, ds} \end{aligned}$$

for  $t$  sufficiently large, i.e.,

$$\left( \frac{\int_T^t H^{-1}(s)u^2(s) \, ds}{\int_T^t H^{-1}(s) \, ds} \right)^{\frac{1}{2}} \geq \frac{1}{2} pK_1 \frac{\int_T^t H^{-1}(s) \left( \int_T^s H^{-1}(\tau)u^2(\tau) \, d\tau \right) \, ds}{\int_T^t H^{-1}(s) \, ds}.$$

Denote  $S(t) := \int_T^t H^{-1}(s) \left( \int_T^s H^{-1}(\tau)u^2(\tau) \, d\tau \right) \, ds$ . Then

$$\left( \frac{S'(t)H(t)}{\int_T^t H^{-1}(s) \, ds} \right)^{\frac{1}{2}} \geq \frac{1}{2} pK_1 \frac{S(t)}{\int_T^t H^{-1}(s) \, ds}.$$

By simple calculation we obtain

$$\frac{S'(t)}{S^2(t)} \geq \frac{1}{4} p^2 K_1^2 \frac{H^{-1}(t)}{\int_T^t H^{-1}(s) \, ds}.$$

Integrating from  $T_1 > T$  to  $t$  we get

$$\frac{1}{S(T_1)} > \frac{1}{S(T_1)} - \frac{1}{S(t)} \geq \frac{1}{4} p^2 K_1^2 \ln \left( \int_{T_1}^t H^{-1}(s) \, ds \right) \rightarrow \infty$$

for  $t \rightarrow \infty$ , and this is a contradiction with the convergence of  $\int^\infty H^{-1}u^2 \, dt$ .  $\square$



For easier manipulation with certain terms in the subsequent parts of this paper, let us denote

$$L(t) := \frac{\int_T^t H^{-1}(s) \left( \int_T^s (c(\tau) - \tilde{c}(\tau)) h^p(\tau) d\tau \right) ds}{\int_T^t H^{-1}(s) ds}, \quad L(\infty) := \lim_{t \rightarrow \infty} L(t).$$

**Corollary 1.** *Assume that the assumptions of Theorem 1 hold. Let either*

$$(22) \quad L(\infty) = \infty \quad \text{or} \quad -\infty < \liminf_{t \rightarrow \infty} L(t) < \limsup_{t \rightarrow \infty} L(t).$$

Then (1) is oscillatory.

**Proof.** Let  $L(\infty) = \infty$  and suppose that (1) is nonoscillatory. Then (19) holds and by Theorem 1 the integral (17) converges for every solution  $u$  of (11) and hence the limit (18) exists as a finite number, which is a contradiction. The proof of sufficiency of the second condition in (22) is similar.  $\square$

The next theorem is the main result of this paper. It can be seen as a kind of generalization of Hartman-Wintner type criteria.

**Theorem 2.** *Let  $\int^\infty r^{1-q}(t) dt = \infty$ . Suppose that equation (5) is nonoscillatory and let  $h$  be a principal solution of (5) such that*

$$\int^\infty H^{-1}(t) dt = \infty, \quad \lim_{t \rightarrow \infty} r(t)h(t)\Phi(h'(t)) := M > 0,$$

where the function  $H$  is defined by (15).

Further, let  $0 \leq \int^\infty c(t) dt < \infty$  and

$$0 \leq \int^\infty (c(t) - \tilde{c}(t))h^p(t) dt < \infty.$$

If the limit  $L(\infty) < \infty$  exists and

$$\limsup_{t \rightarrow \infty} \frac{\int_T^t H^{-1}(s) ds}{\ln \int_T^t H^{-1}(s) ds} (L(\infty) - L(t)) > \frac{1}{2q},$$

then (1) is oscillatory.

**Proof.** Suppose, by contradiction, that (1) is nonoscillatory. In view of Lemma 2 our assumptions ensure the existence of the finite limit

$$(23) \quad \lim_{t \rightarrow \infty} \frac{w(t)}{w_h(t)} = 1,$$

where  $w$  is a solution of the Riccati equation (6) corresponding to (1) and  $w_h = r\Phi(h')/\Phi(h)$  the solution of the Riccati equation corresponding to (5). Let us investigate the behavior of the function  $P(u, v)$ ,

$$P(u, v) = \frac{u^p}{p} - uv + \frac{v^q}{q} = u^p \left( \frac{1}{q} \frac{v^q}{u^p} - vu^{1-p} + \frac{1}{p} \right) = u^p Q(vu^{1-p}),$$

where  $Q(x) = q^{-1}x^q - x + p^{-1} \geq 0$  and  $Q(1) = 0$ . By L'Hospital's rule (used twice) we have

$$\lim_{x \rightarrow 1} \frac{Q(x)}{(x-1)^2} = \frac{q-1}{2}.$$

Hence, for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$(24) \quad -\varepsilon \leq \frac{Q(x)}{(x-1)^2} - \frac{q-1}{2} \leq \varepsilon$$

for  $x$  satisfying  $|x-1| < \delta$ , and inequality (24) can be rewritten as

$$\left( \frac{q-1}{2} - \varepsilon \right) (x-1)^2 \leq Q(x) \leq \left( \frac{q-1}{2} + \varepsilon \right) (x-1)^2.$$

For  $x = vu^{1-p}$  we have

$$\left( \frac{q-1}{2} - \varepsilon \right) (vu^{1-p} - 1)^2 \leq Q(vu^{1-p}) \leq \left( \frac{q-1}{2} + \varepsilon \right) (vu^{1-p} - 1)^2,$$

which is for  $u \neq 0$  equivalent to

$$u^p \left( \frac{q-1}{2} - \varepsilon \right) (vu^{1-p} - 1)^2 \leq P(u, v) \leq u^p \left( \frac{q-1}{2} + \varepsilon \right) (vu^{1-p} - 1)^2.$$

By virtue of (23) there exists  $T_1$  such that  $|w/w_h - 1| < \delta$  for  $t \geq T_1$  and hence for  $u = \Phi^{-1}(w_h(t))$ ,  $v = w(t)$  we have

$$w_h^q \left( \frac{q-1}{2} - \varepsilon \right) \left( \frac{w}{w_h} - 1 \right)^2 \leq P(\Phi^{-1}(w_h), w) \leq w_h^q \left( \frac{q-1}{2} + \varepsilon \right) \left( \frac{w}{w_h} - 1 \right)^2.$$

From the definition of  $w_h$  we get

$$\begin{aligned} h^{2p-2}(t)r^{-1}(t)(h'(t))^{2-p} \left( \frac{q-1}{2} - \varepsilon \right) (w(t) - w_h(t))^2 &\leq r^{1-q}(t)h^p(t)P(\Phi^{-1}(w_h), w) \\ &\leq h^{2p-2}(t)r^{-1}(t)(h'(t))^{2-p} \left( \frac{q-1}{2} + \varepsilon \right) (w(t) - w_h(t))^2, \end{aligned}$$

which, in terms of  $u = (w - w_h)h^p$  and  $H = rh^2|h'|^{p-2}$ , yields

$$(25) \quad \left( \frac{q-1}{2} - \varepsilon \right) \frac{u^2(t)}{H(t)} \leq r^{1-q}(t)h^p(t)P(\Phi^{-1}(w_h), w) \leq \left( \frac{q-1}{2} + \varepsilon \right) \frac{u^2(t)}{H(t)}.$$

As (1) and (5) are nonoscillatory, the modified Riccati equation (11) holds and by its integration and using the fact that  $\int^\infty r^{1-q}h^p P(\Phi^{-1}(w_h), w) < \infty$  (which follows from Lemma 2), we get

$$u(t) = u(T) - \int_T^t (c(s) - \tilde{c}(s))h^p(s) ds - p \int_t^T r^{1-q}(s)h^p(s)P(\Phi^{-1}(w_h), w) ds,$$

hence

$$\begin{aligned} u(t) &= u(T) - p \int_T^\infty r^{1-q}(t)h^p(t)P(\Phi^{-1}(w_h), w) dt \\ &\quad + p \int_t^\infty r^{1-q}(s)h^p(s)P(\Phi^{-1}(w_h), w) ds \\ &\quad - \int_T^t (c(s) - \tilde{c}(s))h^p(s) ds. \end{aligned}$$

Using (21), we get in view of the definition of  $L(\infty)$  and (25)

$$u(t) \geq L(\infty) + \left(\frac{q}{2} - p\varepsilon\right) \int_t^\infty \frac{u^2(s)}{H(s)} ds - \int_T^t (c(s) - \tilde{c}(s))h^p(s) ds,$$

which implies (suppressing the integration variable)

$$\begin{aligned} \int_T^t H^{-1}u \geq \int_T^t L(\infty)H^{-1} + \frac{q}{2} \int_T^t H^{-1} \int_s^\infty \frac{u^2}{H} - \int_T^t H^{-1} \int_T^s (c - \tilde{c})h^p \\ - p\varepsilon \int_T^t H^{-1} \int_s^\infty \frac{u^2}{H}, \end{aligned}$$

and hence

$$\begin{aligned} \int_T^t L(\infty)H^{-1}(s) ds - \int_T^t H^{-1}(s) \int_T^s (c(\tau) - \tilde{c}(\tau))h^p(\tau) d\tau ds \\ \leq \int_T^t \frac{1}{H(s)}u(s) ds - \frac{q}{2} \int_T^t \frac{1}{H(s)} \int_s^\infty \frac{u^2(\tau)}{H(\tau)} d\tau ds + p\varepsilon \int_T^t \frac{1}{H(s)} \int_s^\infty \frac{u^2(\tau)}{H(\tau)} d\tau ds. \end{aligned}$$

Using the definition of  $L(t)$  on the left-hand side and integrating by parts on the right-hand side of the last inequality, we have

$$\begin{aligned} (L(\infty) - L(t)) \int_T^t H^{-1}(s) ds \\ \leq \int_T^t H^{-1}(s)u(s) ds - \frac{q}{2} \left[ \int_s^\infty \frac{u^2(\tau)}{H(\tau)} d\tau \cdot \int_T^s H^{-1}(\tau) d\tau \right]_T^t \\ - \frac{q}{2} \int_T^t \left( \frac{u^2(s)}{H(s)} \int_T^s H^{-1}(\tau) d\tau ds \right) + p\varepsilon \int_T^t H^{-1}(s) \int_s^\infty \frac{u^2(\tau)}{H(\tau)} d\tau ds \end{aligned}$$

and

$$\begin{aligned}
& (L(\infty) - L(t)) \int_T^t H^{-1}(s) \, ds \\
& \leq \int_T^t \frac{H^{-1}(s)}{\int_T^s H^{-1}(\tau) \, d\tau} \left( u(s) \int_T^s H^{-1}(\tau) \, d\tau - \frac{q}{2} \left( u(s) \int_T^s H^{-1}(\tau) \, d\tau \right)^2 \right) ds \\
& \quad - \frac{q}{2} \int_t^\infty \frac{u^2(s)}{H(s)} \, ds \int_T^t H^{-1}(s) \, ds + p\varepsilon \int_T^t H^{-1}(s) \int_s^\infty \frac{u^2(\tau)}{H(\tau)} \, d\tau \, ds
\end{aligned}$$

and by virtue of the inequality  $\alpha - \frac{1}{2}q\alpha^2 \leq \frac{1}{2}q^{-1}$  for  $\alpha = u \int^s H^{-1}$  we get

$$\begin{aligned}
(L(\infty) - L(t)) & \leq \frac{1}{2q} \frac{\ln \int_T^t H^{-1}(s) \, ds}{\int_T^t H^{-1}(s) \, ds} - \frac{q}{2} \int_t^\infty \frac{u^2(s)}{H(s)} \, ds \\
& \quad + p\varepsilon \frac{\int_T^t H^{-1}(s) \int_s^\infty H^{-1}(\tau) u^2(\tau) \, d\tau \, ds}{\int_T^t H^{-1}(s) \, ds}.
\end{aligned}$$

From Theorem 1 we obtain that  $\int_t^\infty H^{-1} u^2 < \infty$  and thus

$$\limsup_{t \rightarrow \infty} \frac{\int_T^t H^{-1}(s) \, ds}{\ln \int_T^t H^{-1}(s) \, ds} (L(\infty) - L(t)) \leq \frac{1}{2q} + p\varepsilon \int_t^\infty \frac{u^2(s)}{H(s)} \, ds.$$

As  $\lim_{t \rightarrow \infty} w/w_h = 1$ ,  $\varepsilon$  and also the last term of the above inequality are arbitrarily small and we have a contradiction with our assumption.  $\square$

**Corollary 2.** Let  $r(t) \equiv 1$ ,  $\tilde{c} = \tilde{\gamma}/t^p$  where  $\tilde{\gamma} = ((p-1)/p)^p$ , i.e., (5) is the generalized Euler equation with the critical coefficient

$$(26) \quad (\Phi(y'))' + \frac{\tilde{\gamma}}{t^p} \Phi(y) = 0.$$

Let  $\int_t^\infty c(s) \, ds \geq 0$  for large  $t$  and

$$0 \leq \int_t^\infty \left( c(s) - \frac{\tilde{\gamma}}{s^p} \right) s^{p-1}(s) \, ds < \infty.$$

If, for  $T$  sufficiently large, the limit

$$L(\infty) = \lim_{t \rightarrow \infty} \frac{\int_T^t s^{-1} \int_T^s (c - \tilde{\gamma}/\tau^p) \tau^{p-1} \, d\tau \, ds}{\ln |t/T|} < \infty$$

exists and

$$\limsup_{t \rightarrow \infty} \frac{\ln |t/T|}{\ln \ln |t/T|} \left( L(\infty) - \frac{\int_T^t s^{-1} \int_T^s (c - \tilde{\gamma}/\tau^p) \tau^{p-1} d\tau ds}{\ln |t/T|} \right) > \frac{1}{2q},$$

then (1) is oscillatory.

**P r o o f.** The function  $h(t) = t^{(p-1)/p}$  is the principal solution of (26) (see [9]),

$$\lim_{t \rightarrow \infty} h(t)\Phi(h'(t)) = \lim_{t \rightarrow \infty} t^{(p-1)/p} \left( \frac{p-1}{p} t^{-1/p} \right)^{p-1} = \left( \frac{p-1}{p} \right)^{p-1}$$

and

$$\int^{\infty} \frac{dt}{h^2(t)(h'(t))^{p-2}} = \left( \frac{p}{p-1} \right)^{p-2} \int^{\infty} \frac{dt}{t} = \infty.$$

The statement follows from Theorem 2. □

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