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Spectrum Analysis of LTI Continuous-Time Systems With Constant Delays: A Literature Overview of Some Recent Results

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ABSTRACT In recent decades, increasingly intensive research attention has been given to dynamical systems containing delays and those affected by the after-effect phenomenon. Such research covers a wide range of human activities and the solutions of related engineering problems often require interdisciplinary cooperation. The knowledge of the spectrum of these so-called time-delay systems (TDSs) is very crucial for the analysis of their dynamical properties, especially stability, periodicity, and dumping effect. A great volume of mathematical methods and techniques to analyze the spectrum of the TDSs have been developed and further applied in the most recent times. Although a broad family of nonlinear, stochastic, sampled-data, time-variant or time-varying-delay systems has been considered, the study of the most fundamental continuous linear time-invariant (LTI) TDSs with fixed delays is still the dominant research direction with ever-increasing new results and novel applications. This paper is primarily aimed at a (systematic) literature overview of recent (mostly published between 2013 to 2017) advances regarding the spectrum analysis of the LTI-TDSs. Specifically, a total of 137 collected articles–which are most closely related to the research area–are eventually reviewed. There are two main objectives of this review paper: First, to provide the reader with a detailed literature survey on the selected recent results on the topic and Second, to suggest possible future research directions to be tackled by scientists and engineers in the field.

INDEX TERMS Delay systems, eigenvalues and eigenfunctions, literature review, stability.

I. INTRODUCTION

Systems evincing delays or after-effect phenomenon emerge in a wide range of human activities and can be observed in many applications including but not limited to engineering, economy and biology. For instance, the cognitive delays represented by past states can be observed in predator-pray models [1] which are very early real-life-inspired time-delay systems (TDSs). The human nervous system represents another common biological example. The associated cognition-to-reaction delay may cause balancing problems [2], [3] or traffic jams [4]. The effects of delays on the physiological systems were analyzed in [5]. Engineers from various industrial branches must deal with the after-effect phenomena in daily practice, such as delays in metallurgic processes [6], delays in the distribution of heat or power [7], [8], delays incurred by the mass flow in the production of sugar [9], etc. In machining processes, the milling dynamics models are usually represented by delay differential equations [10], and the machine tool chatters can be modeled by the so-called regenerative delays [11]. All in all, the modern world is full of various networks, with their different components communicating with one another, the processes of which are affected by the so-called communication latencies inevitably [12], [13].

Because of a very high interdependence with the everyday routine and practice, TDSs have attracted researchers and engineers since the nascence of modern systems and control theory. This is supported by the fact that the existence of delays may cause much worse dynamical properties of the system (especially, their stability and periodicity), and the attractiveness also increases because of that the inclusion of delays often leads to more realistic (infinite-dimensional) models but it significantly complicates the analysis and synthesis [14]. In the recent decades, an increasing number of international meetings and research articles have dealt with

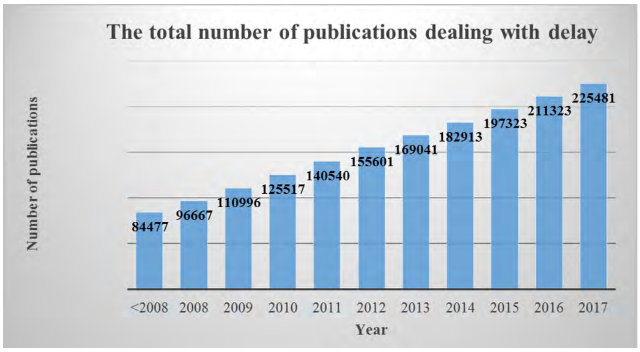

FIGURE 1. **The total number of recently published research articles dealing with *delay(s)* listed in Scopus and WoS indexing databases.**

TDSs; it is worth noting that some important recent books summarize the up-to-date knowledge [15]–[22]. In addition, the top world's leading scientific, research and development publication indexing databases, Elsevier's Scopus and Web of Science (WoS) by Clarivate Analytics, would yield over 200 thousand results if *delay* is included in the searching title or keyword. Fig. 1 shows that the total number of such publications has been increasing over the past ten years, with an approximate annual increment of 14,000 publications. Therefore, it is almost impossible nowadays to address all the most recent advances in the field by only one person.

The knowledge of the loci of the characteristic roots, i.e. system poles, constitutes crucial information about the systems' stability and dynamics. It is well-known that a TDS has an infinite number of characteristic roots, i.e. its spectrum is infinite, and therefore the complete image of their loci is very hard to analyze, if possible at all. The methods and techniques of the spectral analysis of the TDSs deal with the aforementioned negative effects of the delays on the overall performance including stability and dynamics, and they have yielded many practical impacts and conclusions. Although researchers have found many new results on a variety of complex systems with different kinds of delays, including but not limited to time-invariant, time-variant, non-linear, chaotic or discrete-time ones, even more studies have been published in the field of continuous linear time-invariant (LTI) TDSs which represents the most common yet still a very attractive class of TDSs.

In spite of the relative simplicity of LTI-TDS models, an enormous number of books, conferences and journal papers indicates that there are many unsolved problems related to these systems. For example, the very famous work of Richard [23] provided an overview of various modeling, stability analysis and control approaches for LTI-TDSs and discussed some open problems as well, e.g. the control using delayed information.

In addition, Gu and Niculescu [24] presented a broad overview of the stability and control of TDSs concerning practical problems and engineering applications. Various research related to literature overview in the field of TDSs has been done under different names with different aims. For instance, a delay jitter problem in the packet network based telephony was discussed in details in [25] where an overview of various attempting methods was also provided. Wang *et al.* [26] provided a survey on recent progress in the measurement of two types of the end-to-end latencies (round trip delay and one-way delay) over the Internet. Chen *et al.* [27] presented a systematic overview of the time-delay-estimation algorithms ranging from the simple cross-correlation method to the advanced blind channel identification based techniques. Fruchard and Schäfke [28] provided an overview of the problem of bifurcation delay from its appearance in France at the end of the eighties to the most recent contributions. A research focused on the performance of domestic and international transport and logistics systems as perceived by Chinese importers and exporters was provided by Zhang and Figliozzi [29] along with a broad literature review of the meteoric logistics industrial development in China. A survey on the virtual-environment-based and bilateral teleoperations of space robots with time delay effects was addressed in [30]. Wang [31] presented a review of the recent advances in delay-time-based maintenance modeling, which is one of the mathematical techniques for optimizing inspection planning and related problems. A step-by-step introduction to the notion of time-delay in classical and quantum mechanics, aiming at clarifying its foundation at a conceptual level, was made by Sassoli de Bianchi [32]. Cui *et al.* [33] gave a survey on several major systematic approaches in dealing with delay-aware control problems, namely the equivalent rate constraint approach, the Lyapunov stability drift approach, and the approximate Markov decision process approach using stochastic learning. The research of Wang *et al.* [34] focused on a review of some design and tuning methods of active disturbance rejection control methodology for TDS with its applications as well. Flunkert *et al.* [35] provided an overview of the effect of delayed coupling and feedback on dynamical systems. A comprehensive and excellent survey on stochastic hybrid systems by Teel *et al.* [36] addressed results on the stability of delayed ones as well. Similarly, Tao [37] included TDSs into his overview of some fundamental theoretical aspects and technical issues of the multivariable adaptive control. Doudou *et al.* [38] provided a comprehensive review and taxonomy of the state-of-the-art synchronous medium access control protocols with respect to sender-receiver latencies. Lu and Shen [39] provided an overview of the development in the area of scaling laws for the throughput capacity and the delay in wireless networks. Liu and Yang [40] summarized the progress of grey system research from 2004 to 2014 including the problems of robust stability for grey stochastic time-delay systems of neutral type, distributed-delay type and neutral distributed-delay type. An overview of the basic results and methods for the stability investigations of higher-order autonomous linear difference equations, with a special emphasis on delay difference equations was published by Čermák [41]. Zhu *et al.* [42] presented a variety of stability conditions for TDSs with varying delays in the form of scaled small-gain conditions. In the broad introduction in [43], Liao *et al.* summarizes the recent advances in optimal, robust and preview control for systems with input delays. Mahmoud [44] provided an overview of the research investigations in the field of networked control systems (NCSs) that include discussions on addressing communication delays. A similar topic was reviewed in [45], where the authors focused on time-delay fuzzy-model-based nonlinear NCSs as well. To name just one publication example from medicine, Wechkunanukul *et al.* [46] provided a summarized review on a range of countries describing the differences in time taken to seek medical care for chest pain and the factors which contribute to delay times. Besides journal review articles, several books summarizing the up-to-date knowledge about TDSs [15], [16], [19], [47], and those representing a compiled set of selected papers from the most important control conferences and specialized workshops within the research area [17], [18], [20]–[22] have been published as well.

Currently, a family of LTI-TDSs is lacking a holistic literature overview that covers and summarizes most of the spectral analysis techniques for systems with constant (known or unknown) delays. This survey paper aims to provide the reader with a literature overview of such recently published results. The exigency of such an up-to-date survey can be considered as the greatest contribution of this work. However, because of a huge number of published results in this area it does not allow to cover all the ideas, therefore this study does not claim to be exhaustive. It should be noted that this study is primarily not interested in control-oriented studies, and that the reader are referred to the cited literature for more details – especially, regarding complete mathematical issues.

Throughout the paper, $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ denote the set of, respectively, complex numbers, real numbers, integers and non-negative integers. The closed unit disk, the unit circle and the imaginary axis are denoted as $\mathbb{D}$, $\mathbb{T}$, and $\mathbb{C}^0$, respectively. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space (a column vector), $\mathbb{R}^{n\times m}$ is the set of all real-valued matrices of the dimension $n \times m$. For $s \in \mathbb{C}$, $\mathrm{Re}(s)$ and $\mathrm{Im}(s)$ denote, respectively, the real and imaginary parts of s; $\mathbb{C}_0^- := \{s \in \mathbb{C} | \mathrm{Re}(s) < 0\}$, $\mathbb{C}^+ = \mathbb{C} \backslash \mathbb{C}_0^-$, the set of real polynomials is denoted as $\mathbb{R}[s]$. The superscript T stands for the matrix transpose. For $F(s) : s \in \mathbb{C} \mapsto F \in \mathbb{C}$, define sets

$$H_2 := \left\{ F(s) : \int_{-\infty}^{\infty} F(\mathrm{j}\omega)\, \overline{F(\mathrm{j}\omega)} \mathrm{d}\omega < \infty \right\},$$
$$H_\infty := \left\{ F(s) : \|F(s)\|_\infty := \sup_{s\in\mathbb{C}^+} |F(s)| < \infty \right\},$$

and let

$$L_p := \left\{ f(t) : \left(\int_0^t |f(t)|^p \mathrm{d}t \right)^{1/p} < \infty, p \in \mathbb{N} \right\}$$

for $f(t) : t \in \mathbb{R} \mapsto f \in \mathbb{R}$. We use $L(\cdot)$ for the Laplace transform of $(\cdot)$. The unit matrix of the dimension n is denoted as $\mathbf{I}_n$.

The rest of the paper is organized as follows: In the preliminary Section II, the research field of LTI-TDSs is specified and their basic spectral and stability properties are introduced. Section III concisely explains the methodology of the literature review used herein the study. Section IV provides a brief overview of some earlier important and famous methods in the field of the spectrum analysis of LTI-TDSs and related stability issues. The main contribution of the paper is given in Sections V and VI. In Section V, a detailed review of recently published papers with a theoretical contribution to pole loci is presented. Section VI includes selected recent results on the stability analysis based on the knowledge of the spectrum. In Section VII recent papers on related practical problems and engineering applications are outlined. Section VIII lists our research questions and points out unexplored areas in the field of spectrum analysis of LTI-TDSs aiming at providing directions for the future research. Finally, Section IX concludes the paper.

II. PRELIMINARIES

In this section, LTI-TDSs are defined to determine the field of study; then their essential spectral properties and stability issues related to the eigenvalue spectrum are introduced.

A. LTI-TDS MODEL

A general continuous-time LTI-TDS with constant delays can be formulated by state and output functional differential equations (FDEs) as

$$\begin{aligned}
&\dot{\mathbf{x}}(t) + \sum_{i=1}^{n_H} \mathbf{H}_i \dot{\mathbf{x}}\left(t - \tau_{H,i}\right) + \int_0^L \mathbf{H}_d(\tau)\, \dot{\mathbf{x}}(t-\tau)\, \mathrm{d}\tau \\
&\quad = \mathbf{A}_0 \mathbf{x}(t) + \sum_{i=1}^{n_A} \mathbf{A}_i \mathbf{x}\left(t - \tau_{A,i}\right) + \mathbf{B}_0 \mathbf{u}(t) \\
&\qquad + \sum_{i=1}^{n_B} \mathbf{B}_i \mathbf{u}\left(t - \tau_{B,i}\right) + \int_0^L [\mathbf{A}_d(\tau)\, \mathbf{x}(t-\tau) \\
&\qquad + \mathbf{B}_d(\tau)\, \mathbf{u}(t-\tau)] \mathrm{d}\tau \\
&\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \sum_{i=1}^{n_C} \mathbf{C}_i \mathbf{x}\left(t - \tau_{C,i}\right) \\
&\qquad + \int_0^L \mathbf{C}_d(\tau)\, \mathbf{x}(t-\tau)\, \mathrm{d}\tau
\end{aligned} \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$, $\mathbf{y}(t) \in \mathbb{R}^l$, stand for the vector of state variables, inputs and outputs, respectively, $\dot{\mathbf{x}}(t) = \mathrm{d}\mathbf{x}(t)/\mathrm{d}t$, $\mathbf{A}_i$, $\mathbf{A}_d(\tau)$, $\mathbf{B}_i$, $\mathbf{B}_d(\tau)$, $\mathbf{C}$, $\mathbf{C}_d(\tau)$, $\mathbf{H}$, $\mathbf{H}_d(\tau)$ are real-valued matrices of compatible dimensions, $0 < \tau_{\cdot,1} < \tau_{\cdot,2} < \ldots \leq L$ express lumped (point-wise) delays and convolution integrals characterize *distributed delays*. If $\tau_{\cdot,i} = n_{\cdot,i} h$ for all i where $n_{\cdot,i} \in \mathbb{N}$, with the base delay, $h \in \mathbb{R}$, delays are called *commensurate*. The initial condition is a function segment $\mathbf{x}(\theta) = \varphi(\theta), \theta \in [-L, 0]$, $\varphi \in \ell([-L, 0], \mathbb{R}^n)$, where ℓ means the Banach space of continuous-time functions mapping $\theta \in [-L, 0]$ into $\mathbb{R}^n$ equipped with the supreme norm $\|\cdot\|_s$. For $t \geq 0$, denote $\mathbf{x}_t = \mathbf{x}_t(\theta, \varphi) = \mathbf{x}(t + \theta), \theta \in [-L, 0]$ for the initial data φ. If $\exists i$ so that

$\mathbf{H}_i \neq \mathbf{0}$ or $\mathbf{H}_d(\tau) \neq \mathbf{0}$, the system is of *neutral type*; otherwise, it is of *retarded type*. For the sake of this review, retarded and neutral systems without distributed delays are abbreviated as RTDSs and NTDSs, respectively, and those with (non-lumped) delay distribution let be dRTDSs and dNTDSs, respectively.

Note that an LTI-TDS can also be represented in the context of the Hilbert space as follows. The homogeneous state equation has the form $\dot{\mathbf{x}}(t) = \widehat{\ell}(t)\mathbf{x}_t,\ t \geq 0$ where $\widehat{\ell}(t) : \mathbb{R}^+ \times \mathcal{X} \to \mathbb{R}^n, \mathcal{X} = \mathcal{H}([-L,0],\mathbb{R}^n)$ with $\mathcal{X}$ being the Hilbert space of continuous functions on the interval $[-L, 0]$. For further details, the reader is referred e.g. to [48].

By using the Laplace transform, one can construct the characteristic function

$$\Delta(s) = \det\begin{bmatrix} s\Big(\mathbf{I} + \sum_{i=1}^{n_H}\mathbf{H}_i\exp\left(-s\tau_{H,i}\right) \\ +\int_0^L \mathbf{H}_d(\tau)\exp(-s\tau)\,\mathrm{d}\tau\Big) \\ -\mathbf{A}_0 - \sum_{i=1}^{n_A}\mathbf{A}_i\exp\left(-s\tau_{A,i}\right) \\ -\int_0^L \mathbf{A}_d(\tau)\exp(-s\tau)\,\mathrm{d}\tau \end{bmatrix} \quad (2)$$

For RTDSs and NTDSs, (2) is a *quasipolynomial*; however, due to distributed delays it takes a form of a quasipolynomial fraction where some numerator/denominator roots can be mutually algebraically canceled. System*poles* (eigenvalues or characteristic values) constituting the spectrum Σ of (1) satisfy

$$\Sigma := \{s : \Delta(s) = 0\}. \quad (3)$$

Let the numerator (if applicable) of $\Delta(s)$ be denoted as $\bar{\Delta}(s) := \sum_{i=0}^{n_\Delta} d_i(s)s^i$ where d_i are exponential polynomials and $n_\Delta \geq n$. Then $d_{n_\Delta}(s) = d_{n_\Delta,0} + \sum_{i=1}^{\tilde{n}} d_{n_\Delta,i}\exp(-s\tilde{\tau}_i)$ with $d_{n_\Delta,\cdot} \in \mathbb{R}$, $\tilde{n} \geq n_H$, $\tilde{\tau}_i \leq L + \sum_{i=1}^{n_H}\tau_{H,i}$ constitutes the *associated characteristic exponential polynomial*. The *essential spectrum* of a (d)NTDS reads

$$\Sigma_{ess} := \left\{s : d_{n_\Delta}(s) = 0\right\}. \quad (4)$$

B. BASIC SPECTRAL PROPERTIES

Let us introduce some very basic properties of Σ, Σ_{ess} as follows, leaving proofs to references:

Proposition 1 [14]–[16]: For a (d)RTDS it can be deduced that:

(i) If the numerator of $\bar{\Delta}(s)$ is a quasipolynomial not a polynomial, then $|\Sigma| = \infty$;

(ii) All $s_k \in \Sigma$ are isolated;

(iii) There are only finitely many characteristic roots $s_k \in \Sigma$ in the strip $\{\beta_1 < \mathrm{Re}(s_k) < \beta_2\} \subset \mathbb{C}$;

(iv) For any $\beta \in \mathbb{R}$ with $\beta < 0$, only finitely many poles are located in the half-plane $\mathrm{Re}s > \beta$, while infinitely many ones are located to the left-hand side of $\mathrm{Re}s = \beta$;

(v) Isolated poles behave continuously and smoothly with respect to $\boldsymbol{\tau}$ and parameters on $\mathbb{C}$.

Proposition 2 [14], [24], [48], [51]: For a (d)NTDS the following statements are true:

(i) Let $s_k \in \check{\Sigma} \subseteq \Sigma$, $s_{e,l} \in \check{\Sigma}_{ess} \subseteq \Sigma_{ess}$ with $|s_{k-1}| < |s_k|$, $\left|s_{e,l-1}\right| < \left|s_{e,l}\right|$. Then for every $\varepsilon > 0$, there exist K, L, such that $\left|s_k - s_{e,l}\right| < \varepsilon$ for $k > K, l > L$. It means that both system and essential spectra constitute vertical strips at high frequencies which converge to each other;

(ii) Define $\gamma := \sup\{\mathrm{Re}(\Sigma_{ess})\}$, then $\lim_{k\to\infty}\mathrm{Im}(s_k) = \infty$ for $|s_{k-1}| < |s_k|$, $s_k \in \check{\Sigma}$ as $\lim_{k\to\infty}\mathrm{Re}(s_k) = \gamma$;

(iii) In the half-plane with $\mathrm{Re}(s) > \gamma$, there may lie infinitely many system poles;

(iv) The value of γ is not continuous with respect to $\boldsymbol{\tau}$ (where $\boldsymbol{\tau} \in \mathbb{R}^{n_\tau} > 0$ represents the vector of all system delays;

(v) Define $\tilde{\gamma} := \sup\{\gamma(\tau + \delta\tau) : \forall\|\delta\tau\| < \varepsilon, \varepsilon > 0\}$, the safe upper bound estimation $\bar{c}$ (that is continuous with respect to delays) on $\tilde{\gamma}$ can be calculated from

$$\bar{c} = \left\{c \in \mathbb{R} : \sum_{i=1}^{\tilde{n}}\left|\frac{d_{n_\Delta,i}}{d_{n_\Delta,0}}\right|\exp(c\tilde{\tau}_i) = 1\right\}, \quad (5)$$

see e.g. [52]. Only a finite number of poles is located in the half-plane with $\mathrm{Re}(s) > \bar{c} \geq \tilde{\gamma}$ and they are isolated.

The *spectral abscissa* is the function

$$\alpha(p) := p \mapsto \sup\{\mathrm{Re}\Sigma\} \quad (6)$$

where p is a system parameter (including delay).

Proposition 3 [53]: For $\alpha(\boldsymbol{\tau})$ of a (d)NTDS (1) the following two claims are valid:

(i) The function may be non-smooth and hence not differentiable; e.g. in points with more than one real pole or conjugate pairs with the same maximum real part;

(ii) It is non-Lipschitz; for instance, at points where the maximum real part has multiplicity greater than one.

C. LTI-TDS STABILITY

In this subsection, stability issues of delayed systems, which are closely related to their eigenvalue spectrum, are concisely addressed. Tasks of stability analysis as well as those attempting to guarantee the stable control system constitute the primary problems solved when dealing with the system spectrum.

Definition 1 [14]–[16], [48], [54]: The system (1) (or more precisely, its null solution) is said to be *stable*, if for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\|\varphi\|_s := \max_{\theta\in[-L,0]}\|\varphi(\theta)\| < \delta$ implies that $\|\mathbf{x}(t)\| < \varepsilon$ for any $t \geq 0$ where $\|\cdot\|$ denotes the Euclidean norm. If, moreover, for any $\varphi \in \ell([-L,0],\mathbb{R}^n)$ holds that $\lim_{t\to\infty}\mathbf{x}(t) = 0$, the system is *asymptotically* stable. System (1) is said to be *exponentially* stable if there exist $a > 0, \mu > 0$ such that $\|\mathbf{x}_t(\theta,\varphi)\|_s \leq a\exp(-\mu t)\|\varphi\|_s, \forall t \geq 0$ for all $\varphi \in \ell([-L,0],\mathbb{R}^n)$.

Proposition 4 [14], [15], [24], [48]: A (d)RTDS (1) is

(i) asymptotically and exponentially stable if and only if $\alpha(\cdot) < 0$,

(ii) stable if and only if $\alpha(\cdot) \leq 0$ and for any pole $s_k \in \mathbb{C}^0$ it holds that

$$\mathrm{rank}\begin{bmatrix} s_k\mathbf{I} - \mathbf{A}_0 - \sum_{i=1}^{n_A}\mathbf{A}_i\exp\left(-s_k\tau_{A,i}\right) \\ -\int_0^L\mathbf{A}_d(\tau)\exp(-s_k\tau)\,\mathrm{d}\tau \end{bmatrix} = n - q_k \quad (7)$$

where q_k is the algebraic multiplicity of s_k.

Whereas asymptotic and exponential stability coincide for a RTDS, it is not the same case for a NTDS. The most delicate is the case of poles in one chain of neutral-type given by (ii) in Proposition 2 shaped asymptotically to the imaginary axis yet located in $\mathbb{C}_0^-$ (with no counterpart in $\mathbb{C}^+$)

Proposition 5 [14], [48], [51], [55]: For a (d)NTDS given by (1), the following statements hold:

(i) It is exponentially stable if and only if there exists $\varepsilon > 0$ such that $\alpha(\cdot) < -\varepsilon$;

(ii) If $\alpha(\cdot) < 0$ and there is a critical chain of poles close to $\mathbb{C}^0$, asymptotic stability may or may not occur. For instance, if $\mathbf{H}_1 \neq \mathbf{0}$, $\mathbf{H}_i = \mathbf{H}_d = \mathbf{0}$, $i > 1$, it depends on geometric and algebraic multiplicities of eigenvalues of $\mathbf{H}_1$.

The family of (d)NTDSs is equipped with the following two specific stability issues.

Definition 2 [16], [56]: A (d)NTDS is said to be *formally* stable if

$$\operatorname{rank}\left[\begin{array}{c}\mathbf{I}+\sum_{i=1}^{n_H}\mathbf{H}_i\exp\left(-s\tau_{H,i}\right)\\ +\int_0^L \mathbf{H}_d(\tau)\exp(-s\tau)\,\mathrm{d}\tau\end{array}\right]=n,\quad \forall s\in\mathbb{C}^+. \quad (8)$$

It is said to be *strongly* stable if $\tilde{\gamma} < 0$.

To rephrase the definition, formal stability means that system (1) has only a finite number of poles in $\mathbb{C}^+$, whereas the strong one expresses that it remains formally stable under small delay deviations (see items (iv) and (v) of Proposition 1 and Proposition 2).

Other commonly concerned stability issues, such as BIBO (Bounded-Input Bounded-Output) and H_∞ stabilities cannot be investigated solely from pole loci. Note that, in the single-input single-output (SISO) case – for the simplicity – BIBO means that $|u(t)| < M_1$, $M_1 > 0$ implies $|y(t)| < M_2$, $M_2 > 0$ for $t \geq 0$; or equivalently, the system impulse response $g(t) \in L_1$. The system is H_∞ stable if its transfer function satisfies $G(s) \in H_\infty$ (i.e. it is a bounded analytic function in $\mathbb{C}^+$); or equivalently, $u(t) \in L_2$ implies $y(t) \in L_2$. For instance, let $G_1(s) = 1/(s + s\exp(-s) + 1)$, $G_2(s) = G_1(s)/(s+1)$, $G_3(s) = G_1(s)/(s+1)^4$. All the three transfer functions have poles asymptotically approaching $\mathbb{C}^0$ from the left at infinity. It can be proved that $G_1 \notin H_\infty$, yet $G_2, G_3 \in H_\infty$, and G_1, G_2 are not BIBO stable, unlike G_3 [57].

Definition 3: System (1) is said to be (weakly) *delay-independent stable* (DIS) if it remains stable for any $\boldsymbol{\tau} \in \mathbb{R}^{n_\tau} > 0$. It is said to be*delay-dependent stable* (DDS) if it is stable for a disjoint set $\left\{I_{\tau,i}\right\}$ of regions in the delay space.

Note that DIS and DDS according to Definition 3 are usually considered in terms of exponential or asymptotic stability.

III. LITERATURE REVIEW METHODOLOGY

The purpose of a systematic review is to summarize the best available research on a particular topic. It should transparently collect, analyze and synthesize the results of relevant research. As stated above, the authors do not dare to call this survey as systematic due to the enormous number of results on the addressed topic. However, they attempted to follow the principles of the systematic review. Four steps of the literature review are implemented in this research: planning, searching, screening, and extraction.

In the planning phase, the problem to be addressed is specified in the form of clear research questions. Based on the preliminary section the following research questions are considered here: What is the current status of research on the spectrum analysis for LTI-TDSs with constant delays? What are the open problems in this field?

In early 2018, research papers related to the spectrum analysis for LTI-TDSs with constant delays were searched and collected by the authors. Note that nonlinear, time-varying, stochastic, non-integer-order systems, etc., are not within the scope of this study.

Inclusion and exclusion criteria are defined and applied to the found results in the screening phase. To present up-to-date results, the publication period was determined to be within the last five years (i.e., from 2013 to 2017). It is noteworthy that the authors of this review are by no means intended to claim that earlier results are less important – yet, this paper is primarily aimed at giving an overview of some most recent results. However, some others (out of this period) of critical importance (i.e. those not within the period) are also included when appropriate, and a brief overview of the famous earlier methods is presented as well. A total amount of 137 results covered by the aforementioned period have been finally selected to provide some significant insights into the considered research questions, based on the screening of abstracts and the linkage of particular cited papers and citing sources; the relevance to the topic of this review has been the most important selection criterion. For instance, methods based on Lyapunov stability theory and linear matrix inequalities (LMIs) are mostly out of this scope of this study since they usually provide only the estimate of the exponential decay, i.e. the spectral abscissa, not a deeper insight into the spectrum.

Finally, in the extraction phase, open research questions arising from the analysis of selected papers are concisely introduced and discussed (see Section VIII).

IV. SOME WELL-ESTABLISHED METHODS OF LTI-TDS SPECTRUM ANALYSIS

In this section, we briefly summarize selected significant and famous well-established direct methods on LTI-TDS spectrum analysis and related stability issues.

A. SPECTRUM APPROXIMATION BY SPECTRAL AND PSEUDOSPECTRAL METHODS

This methodology attempts to estimate the infinite-dimensional spectrum by a sufficiently accurate finite-dimensional one by means of a discretized state-space formulation (i.e., the discretization of the so-called solution operator or the infinitesimal generator).

The solution operator $\mathcal{T}(t)$ is defined by the relation

$$\mathcal{T}(t)\varphi = \mathbf{x}_t(\varphi), \quad t \geq 0. \tag{9}$$

The infinitesimal generator $\mathcal{A} : D(\mathcal{A}) \subseteq \ell \to \ell$ of $\mathcal{T}(t)$ has domain

$$D(\mathcal{A}) = \left\{\varphi \in \ell, \varphi' := \mathrm{d}\varphi(\theta)/\mathrm{d}\theta \in \ell : F_l(\varphi, 0) = F_r(\varphi, 0)\right\} \tag{10}$$

where $F_l(\varphi, 0), F_r(\varphi, 0)$ express left-hand and right-hand sides, respectively, of the first equality in (1) where $\mathbf{x}$ is substituted by φ, $t = 0$ and $\mathbf{u}(\cdot) \equiv 0$. The generator acts as $\varphi = \mathcal{A}\varphi, \varphi \in \mathbb{D}(\mathcal{A})$. Then (1) can be treated as an abstract Cauchy problem given by the following operator (ordinary) differential equation

$$\begin{aligned}\dot{\mathbf{x}}_t &= \mathcal{A}\mathbf{x}_t, \quad t \geq 0\\ \mathbf{x}_0 &= \varphi.\end{aligned} \tag{11}$$

It holds that

$$\begin{aligned}s_k &= \frac{1}{t}\ln\mu, \mu \in \sigma\left(\mathcal{T}(t)\right)/\{0\}\\ s_k &\in \sigma(\mathcal{A}).\end{aligned} \tag{12}$$

where $s_k \in \Sigma$, and $\sigma(\cdot)$ denotes the matrix spectrum. Hence, the problem can be transformed into a suitable matrix discretization of the solution operator or its infinitesimal generator.

Many of these methods discretize $\mathbf{x}_t$ by a finite-dimensional approximation in the form of a block vector $\mathbf{X}_t \in \ell_N$ with components $\tilde{\mathbf{x}}_t(\theta_{N,i})$ where ℓ_N is the space of discrete functions defined on the grid $\Omega_N = \left[\theta_{N,i}, i = 0, 1, \ldots, N\right]$, $\theta_{N,0} = 0$, $\theta_{N,i} > \theta_{N,i+1}$, $\theta_{N,\mathbb{N}} = -L$, and $\tilde{\mathbf{x}}_t$ is the approximation of $\mathbf{x}_t$. Then (11) can be approximated as

$$\begin{aligned}\dot{\mathbf{X}}_t &= \mathbf{A}_N\mathbf{X}_t\\ \mathbf{A}_N &\in \mathbb{R}^{n(N+1)\times n(N+1)}.\end{aligned} \tag{13}$$

and the solution operator (9) is usually computed by the following one-step approximation

$$\begin{aligned}\mathbf{X}_{t+\Delta t} &= \mathbf{T}_N(\Delta t)\mathbf{X}_t, \quad \mathbf{T}_N \in \mathbb{R}^{n(N+1)\times n(N+1)}\\ \Delta t &= \theta_{N,i+1} - \theta_{N,i}.\end{aligned} \tag{14}$$

where different techniques are used to determine $\mathbf{A}_N$ or $\mathbf{T}_N(\Delta t)$, see e.g. [14], [58]–[60].

B. FREQUENCY-DOMAIN APPROACHES TO GET FINITE-DIMENSIONAL MODEL REDUCTION

Whenever an LTI model (1) is represented by the transfer function, one may apply a rational approximation of exponential terms to obtain a finite-dimensional model, the spectrum of which can be easily computed [61], [62]. However, some artificial roots appear after the approximation. For instance, the nth order Padé, diagonal Padé, Laguerre and Kautz shift approximations can be expressed as $\exp(-\tau s) \approx p(-s)/p(s)$ where, respectively,

$$\begin{aligned}p(s) &= \sum\nolimits_{k=0}^{n}\binom{n}{k}\frac{(2n-k)!}{(2n)!}(\tau s)^k,\\ p(s) &= \sum\nolimits_{k=0}^{n}\frac{(2n-k)!}{k!(n-k)!}(-s\tau)^k,\\ p(s) &= \lim_{n\to\infty}\left(1+\frac{\tau s}{2n}\right)^n.\\ p(s) &= \left(1+\frac{\tau s}{2n}+\frac{1}{2}\left(\frac{\tau s}{2n}\right)^2\right)^n.\end{aligned}$$

Theses approximations play an essential role in some other delay approximation concepts. For instance, a famous result based on the Trotter-Kato approximation theorem [63] for strongly continuous semigroups was developed in [64]. Using this general framework, two families of particular approximation schemes were constructed. Approximation of the state is done by functions which are piecewise polynomials on a mesh (m-th order splines of deficiency m).

C. LAMBERT W FUNCTION

The use of the Lambert W function yields the analytical solution of the pole loci. However, it is only applicable to systems with commensurate delays. The Lambert function $W(z)$ is a multivalued complex function defined as

$$z = W(z)\exp(W(z)) \tag{15}$$

where its solution constitutes an infinite set of branches and z can be a scalar or a matrix [67]. We refer to $W_k(z)$ as the k-th branch of the Lambert W function of z. The main idea is to express the solution of (3) by means of the Lambert W function. For instance, let the scalar system be $\dot{x}(t) = a_0x(t) + a_1x(t-L)$, then the characteristic equation can be written as $(s-a_0)\exp(sL) = a_1$ which is equivalent to the identity $L(s-a_0) = W(a_1L\exp(-a_0L))$, i.e., $z = a_1L\exp(-a_0L)$, and hence the solution of the characteristic equation reads $s = W(a_1L\exp(-a_0L))/L + a_0$.

D. ARGUMENT PRINCIPLE TECHNIQUE

If the task is to decide on the number N_D of poles located inside a region $D \subset \mathbb{C}$ given by the closed positive Jordan curve φ^+, it holds that

$$N_D = \frac{1}{2\pi\mathrm{j}}\int_{\phi^+}\frac{\Delta'(s)}{\Delta(s)}\mathrm{d}s = \frac{1}{2\pi}\Delta_{\phi^+}\arg\Delta(s), \tag{16}$$

where $\Delta(s)$ is a retarded characteristic quasipolynomial, and $\Delta'(s) = \mathrm{d}\Delta(s)/\mathrm{d}s$ [65]. From (16) it follows that, if $\Delta(0) > 0$, $\Delta(s) \neq 0$ for any $s = \mathrm{j}\omega, \omega \geq 0$, then

$$\mathcal{N} = \frac{n}{2} - \frac{1}{\pi}\Delta_{s=\mathrm{j}\omega,\omega\in[0,\infty)}\arg\Delta(s)$$

where $\mathcal{N}$ means the number of poles in $\mathbb{C}^+$. In [66], the result was extended to NTDSs as follows: Consider a strongly stable system with $\Delta(0) > 0$, $\Delta(s) \neq 0$ for any $s = \mathrm{j}\omega$, $\omega \geq 0$, then (1) is asymptotically stable if and only if

$$\frac{n\pi}{2} - \Phi \leq \Delta \underset{s=\mathrm{j}\omega,\omega\in[0,\infty)}{\arg \Delta(s)} \leq \frac{n\pi}{2} + \Phi,$$
$$\Phi = \arcsin\left(\sum_{i=1}^{\tilde{n}} \left|d_{n_\Delta,i}\right|\right).$$

As clear from Proposition 4 and Proposition 5, exponential stability can effectively be studied by the determination of system (or even delay) parameters such that its spectrum crosses imaginary axis. A bunch of well-established techniques follows. The techniques rely on two important fundamental theoretical results; namely, the associated D-decomposition/τ -decomposition theorem [68], [69], and the continuity property of system eigenvalues with respect to system parameters [70]. By D-decomposition/ τ -decomposition theorem, one can find countably many regions in the parameter space, where in each region the system can possess only a finite number of $\mathcal{N}$. The system is stable for all the points in that region whenever $\mathcal{N} = 0$, and the boundaries that separate these regions are formed by some critical parameter values that impart imaginary-axis poles.

E. REKASIUS SUBSTITUTION

Rekasius [71] proposed the following exact (unlike the Padé one) transformation implemented to calculate the stable-unstable regions of an LTI-TDS

$$\exp(-\tau_i \mathrm{j}\omega) \rightarrow \frac{1 - \mathrm{j}T_i\omega}{1 + \mathrm{j}T_i\omega}, \quad i = 1, 2, \ldots n_\tau \tag{17}$$

where $T_i \in \mathbb{R}$ are so-called pseudo-delays. As a consequence, a new characteristic polynomial in ω parameterized by $\mathbb{T}_i$ is obtained. The substitution (17) is valid and exact if τ complies with

$$\tau_{i,k} = 2\omega^{-1}\left(\tan^{-1}(\omega T_i) + k\pi\right), \quad k \in \mathbb{Z}. \tag{18}$$

That is, every pair satisfying (17) with $s = \mathrm{j}\omega$ is mapped to delays according to (18). These points bisect the delay parameter space into intervals, where in each interval the system is either stable or unstable.

F. ELIMINATION OF TRANSCENDENTAL TERMS IN THE CHARACTERISTIC EQUATION (DIRECT METHOD)

The core idea of the direct method [72] lies in the iterative elimination of exponential terms in $\Delta(s)$ with commensurate delays based on the fact that whenever $\Delta(\mathrm{j}\omega) = 0$, it also holds that $\Delta(-\mathrm{j}\omega) = 0$ (the complex conjugate symmetry of complex roots). Let $\Delta(s) = \sum_{i=0}^{2} d_i(s)\, s^i \exp(-ihs)$, $h > 0$, then construct

$$\begin{aligned}\Delta^{(1)}(s) &:= d_0(-s)\,\Delta(s) - d_2(s)\,\Delta(-s)\exp(-2hs)\\ &= d_0(-s)\,d_0(s) - d_2(-s)\,d_2(s)\\ &\quad + (d_0(-s)\,d_1(s) - d_1(-s)\,d_2(s))\exp(-hs)\\ &=: d_0^{(1)}(s) + d_1^{(1)}(s)\exp(-hs).\end{aligned} \tag{19}$$

The second iteration reads

$$\begin{aligned}\Delta^{(2)}(s) &:= d_0^{(1)}(s)\,d_0^{(1)}(-s) - d_1^{(1)}(s)\,d_1^{(1)}(-s)\\ &=: d_0^{(2)}(s).\end{aligned} \tag{20}$$

Since (20) has no exponential terms and it holds that the elimination procedure preserves the imaginary solutions $s_c = \pm\mathrm{j}\omega_c$ of the original characteristic equation, one can write the following polynomial equation to get the imaginary poles

$$W\left(\omega_c^2\right) = \Delta^{(2)}(\mathrm{j}\omega_c) = 0. \tag{21}$$

The set of potential base delays is then computed by

$$h_k = \omega_c^{-1}\left(\tan^{-1}\left(\frac{\mathrm{Re}\left(d_0^{(1)}(\mathrm{j}\omega_c)/d_1^{(1)}(\mathrm{j}\omega_c)\right)}{\mathrm{Im}\left(-d_0^{(1)}(\mathrm{j}\omega_c)/d_1^{(1)}(\mathrm{j}\omega_c)\right)}\right) + 2k\pi\right), \quad k \in \mathbb{N}_0. \tag{22}$$

The crossing property is then checked by a nonzero RT value calculated via $RT = \mathrm{sgn}\left\{d_0^{(1)}(\mathrm{j}\omega_c)\,\mathrm{d}W\left(\omega_c^2\right)/\mathrm{d}\left(\omega_c^2\right)\right\}$.

G. FREQUENCY SWEEPING

This is very simple yet effective technique to determine stability border in the parameter (or delay) space see e.g. [73]. However, it can be used only if $\Delta(s|\mathbf{p})$ is linear with respect to the unknown parameters $\mathbf{p}$. The leading idea was based on the partition of $\Delta(s, \mathbf{p})|_{s=\mathrm{j}\omega}$ into real and imaginary parts, i.e., $\mathrm{Re}(\Delta(\mathrm{j}\omega, \mathbf{p}))$ and $\mathrm{Im}(\Delta(\mathrm{j}\omega, \mathbf{p}))$, respectively. Then, the common solution of $\mathrm{Re}(\Delta(\mathrm{j}\omega, \mathbf{p})) = \mathrm{Im}(\Delta(\mathrm{j}\omega, \mathbf{p})) = 0$ can be plotted in the parameter space for $\omega \in [0, \omega_{\max}]$, where $\omega_{\max}$ means a particularly selected maximum frequency. The generated plot in the parameter space represents a potential stability boundaries and they determine regions that can be further tested in order to identify the stable ones (for instance, via the D-decomposition procedure).

H. SCHUR-COHN PROCEDURE

This procedure can be used to compute the stability margin for LTI-TDSs with commensurate delays [15], [74]. The original Schur-Cohn procedure computes the determinant of a partitioned matrix

$$S = \begin{pmatrix} \Lambda_1(s) & \Lambda_2(s) \\ \Lambda_2^*(s) & \Lambda_1^*(s) \end{pmatrix}$$

where $\Lambda_1, \Lambda_2 \in \mathbb{C}^{n\times n}[s]$ are appropriate matrices over the ring of polynomial in s, and the asterisk denotes the particular Hermitian.

In order to determine all the imaginary roots of the characteristic quasipolynomial $\Delta(s)$ with the base delay τ_0 and the commensuracy degree n_C, the variable $q = \exp(-\tau_0 s)$ is introduced and $\Delta(s)$ is rewritten as a polynomial $p(s, q)$ in two unknowns over $\mathbb{R}$. Then, the Schur-Cohn criterion solves the problem of computing the values of ω such that $p(\mathrm{j}\omega, q) = 0$ by multiplying $p(\mathrm{j}\omega, q)$ by q^{-i}, $i = 0, 1, \ldots n_C - 1$, and the complex conjugate $\bar{p}(\mathrm{j}\omega, q)$ by q^i, $i = 1, \ldots n_C$. This yields a system of $2n_C$ homogenous equations with the unknowns q^i.

I. KRONECKER SUM AND MATRIX PENCIL METHOD

To introduce these techniques concisely, let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$, then the spectrum of the Kronecker sum $\mathbf{A} \oplus \mathbf{B} = \mathbf{A} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{B}$ consists of all sums in the form of $\lambda + \mu$, where λ, μ belong to the spectrum of $\mathbf{A}, \mathbf{B}$, respectively, and $\mathbf{I}$ is the identity matrix [15], [74]–[76].

For instance, consider the following simple system

$$\dot{\mathbf{x}}(t) = \mathbf{A}_0 \mathbf{x}(t) + \mathbf{A}_1 \mathbf{x}(t - L). \tag{23}$$

If some $s_k, -s_k$ are zeros of $\Delta(s)$ for (23), then s_k belongs to the spectrum of $\mathbf{A}_0 + \mathbf{A}_1 z$, while $-s_k$ lies in the spectrum of $\mathbf{A}_0 + \mathbf{A}_1 z^{-1}$, where $z = \exp(-sL)$; hence

$$\Delta_{ass,1}(z) := \det\left((\mathbf{A}_0 + \mathbf{A}_1 z) \oplus \left(\mathbf{A}_0 + \mathbf{A}_1 z^{-1} \right) \right) = 0. \tag{24}$$

If s_k is a purely imaginary system pole, (24) must hold. The transcription of the Kronecker sum into the matrix pencil form reduces the problem to the existence of the unit circle of generalized eigenvalues of the corresponding matrix pencil. Crossing delays are then given by $\tau_k = \omega^{-1}(\arg(z) + 2k\pi), k \in \mathbb{Z}$.

J. KRONECKER MULTIPLICATION METHOD

The main result that characterizes this framework can be expressed by the following theorem.

Theorem 1 [77]: Let Ω_Π be the set of all eigenvalues of the matrix

$$\mathbf{\Pi} = \begin{pmatrix} \mathbf{0} & \mathbf{\Gamma}_0 \\ -\mathbf{\Lambda}_2 & -\mathbf{\Lambda}_1 \end{pmatrix} \in \mathbb{R}^{n^2 \times n^2} \tag{25}$$

where

$$\begin{aligned} &\mathbf{\Gamma}_0 = \mathbf{I} \otimes \mathbf{I}, \mathbf{\Gamma}_1 = \mathbf{I} \otimes \mathbf{A}_0 - \mathbf{A}_0 \otimes \mathbf{I}, \\ &\mathbf{\Gamma}_2 = \mathbf{A}_1 \otimes \mathbf{A}_1 - \mathbf{A}_0 \otimes \mathbf{A}_0, \\ &\mathbf{\Lambda}_1 = \mathbf{\Gamma}_0^{-1}\mathbf{\Gamma}_1, \quad \mathbf{\Lambda}_2 = \mathbf{\Gamma}_0^{-1}\mathbf{\Gamma}_2. \end{aligned} \tag{26}$$

Then $\Omega \subseteq \Omega_\Pi$ where $\Omega := \mathrm{Im}\left(\Sigma \cap \mathbb{C}^0\right)$.

Last but not least, let us introduce two well-established frameworks from the field that enables to decide about the exponential stability, i.e., whether all characteristic roots are located in $\mathbb{C}_0^-$.

K. EXTENDED HERMITE-BIEHLER THEOREM

The famous work of Bellman and Cooke [78] applied the extension of the Hermite-Biehler theorem to RTDSs based on the earlier work of Pontryagin [79]. This theorem yields the necessary and sufficient conditions for the stability of $\Delta(s)$. Let $\Delta^*(s) = \Delta(s)\exp(sL)$ (having the same roots as $\Delta(s)$), $\Delta_R^*(s) = \mathrm{Re}(\Delta^*(s))$, $\Delta_I^*(s) = \mathrm{Im}(\Delta^*(s))$, then the system is stable if and only if $\Delta_R^*(s)$ and $\Delta_I^*(s)$ have only simple real roots, these roots interlace, and $\left(\Delta_I^*(\omega_0)\right)' \Delta_R^*(\omega_0) - \Delta_I^*(\omega_0)\left(\Delta_R^*(\omega_0)\right)' > 0$ for some $\omega_0 \in (-\infty, \infty)$. The crucial problem is to make sure that $\Delta_R^*(s)$, $\Delta_I^*(s)$ have only real roots, see e.g. [80] for details.

L. CLUSTER TREATMENT OF CHARACTERISTIC ROOTS

The famous Cluster Treatment of Characteristic Roots (CTCR) paradigm was developed more than a decade ago, see e.g. [81], and it originally consists of the following basic steps: First, the characteristic quasipolynomial is transformed to the corresponding polynomial $p(\omega, \mathbf{T})$ including pseudo-delays T_i (via the Rekasius substitution). Second, the eventual polynomial is subjected to the Routh array scheme to get the so-called potential stability switching hypersurfaces (or, the kernel curves in the 2D delay space) defined as

$$\wp_0(\boldsymbol{\tau}) := \Omega_\tau \left[\boldsymbol{\tau} : \Delta(s, \tau) = 0, s = \mathrm{j}\omega, \omega \in \Omega_c \right] \tag{27}$$

where Ω_τ represents all kernel points in the delay space and Ω_c means the set of all possible corresponding crossing frequencies. The necessary crossing condition is that the root tendency $RT := \mathrm{sgn}\left\{ \mathrm{Re}\left(\mathrm{d}s/\mathrm{d}\tau_i|_{s=\mathrm{j}\omega} \right) \right\}, i = 1, 2, .., n_\tau$, is nonzero, where n_τ denotes the number of independent delays in $\Delta(s)$. As the third step, the kernel curves (or, potential stability switching hypersurfaces) together with the crossing frequencies give rise to the so-called offspring hypersurfaces $\wp_{off}(\boldsymbol{\tau})$ which are generated based on the knowledge that whenever there exists an imaginary pole $s_c = \pm \mathrm{j}\omega_c$ for some $\boldsymbol{\tau}_0$, the same pole exists for $\tau_i = \tau_{0,i} + 2\pi k_i/\omega_c$, $\tau_{0,i} - 2\pi/\omega_c \leq 0$, $i = 1, 2, .., n_\tau$; $k_i \in \mathbb{N}$, as well. Finally, the D-subdivision method [68] is deployed to determine the number of unstable poles on the right half-plane, starting from the delay-free case.

V. OVERVIEW OF RECENT THEORETICAL RESEARCH ON POLE LOCI CALCULATION, COMPUTATION AND APPROXIMATION

This section is dedicated to the review of recent theoretical research on the analysis of the eigenvalue (characteristic roots) spectrum Σ of LTI-TDSs with constant parameters and delays. Methods for the root loci computation or approximation and those determining a part of Σ within the specified region in $\mathbb{C}$ are presented.

Methods for the computation of the characteristic roots covered herein can be divided into the following categories:

(i) Numerical methods based on the approximation of the solution operator associated with the system (1) or its infinitesimal generator via spectral (or pseudospectral) method;

(ii) Semidiscretization and full-discretization methods;

(iii) Numerical integration and differential quadrature methods;

(iv) Contour integral method;

(v) Lambert W function;

(vi) Special numerical, semi-analytic and analytic methods.

A. SPECTRAL AND PSEUDOSPECTRAL METHODS

In the literature, Wu and Michiels [60] summarized the methodology introduced in (9)-(14) to get $\mathbf{A}_N$ concerning the computation of all characteristic roots in a given right half-plane and they also provided a procedure for the automatic selection of N; Chebyshev polynomials of the first kind were utilized to approximate $\mathbf{x}_t$ by $\tilde{\mathbf{x}}_t$ – the so-called pseudospectral collocation (PsC) method is then obtained (the term pseudospectral means that the solution is approximated in a finite dimensional subspace).

Another point of view was presented by Lehotzky and Insperger [48] where a detailed comparison of a family of weighted residual-type methods to approximate the operator differential equation (11) in the Hilbert space (see after (1)) was provided. In more detail, let the solution of (11) be approximated by $\tilde{\mathbf{x}}_t(\theta) = \sum_{j=1}^{N_a} \phi_j(\theta)\,\mathbf{a}_j(t)$ where $\theta \in [-L, 0]$, and $\mathbf{a}_j(t)$ are unknown variables spanned by the basis $\{\phi_j\}_{j=1}^{N_a}$, then the residual function reads $\mathbf{r}_t(\theta) = \dot{\tilde{\mathbf{x}}}_t - \tilde{\mathbf{x}}'_t$. The objective is to determine $\mathbf{a}_j(t)$ to get $\tilde{\mathbf{x}}_t \to \mathbf{x}_t$ as close as possible. Methods of weighted residuals yield $\langle \mathbf{r}_t, \psi_j \rangle := \int_{-L}^{0} \mathbf{r}_t(\theta)\,\psi_j(\theta)\mathrm{d}\theta = 0, j = 1, 2, \ldots, N_a$ [48], [82], where $\psi_j(\theta)$ are test functions, which can also be represented in a matrix form as

$$\mathbf{N}\dot{\mathbf{a}}(t) = \mathbf{M}\mathbf{a}(t). \tag{28}$$

where $\mathbf{a}(t) = \left[\mathbf{a}_j(t)\right]_{j=1}^{N_a}$, $\mathbf{N}, \mathbf{M} \in \mathbb{R}^{nN_a \times nN_a}$, and entries of these matrices include inner products $\langle \phi_j, \psi_j \rangle$. In the so-called pseudospectral tau approximation (PsT), the rough solution segment is given by

$$\tilde{\mathbf{x}}_t(\theta) = \sum_{j=1}^{N_a} \phi_j(\theta)\,\mathbf{x}_t(\theta). \tag{29}$$

where $\phi_j(\theta)$ are Lagrange base polynomials, i.e. $\phi_j\left(\theta_{N,i}\right) = 1$ for $i = j$, else $\phi_j\left(\theta_{N,i}\right) = 1$. The barycentric formula of Lagrange polynomials used in [48] is more numerically stable and derivatives of ϕ_j are then less complicated. The PsC method introduced above (also called the Chebyshev spectral continuous-time approximation) reduces the error $\tilde{\mathbf{x}}_t(\theta) - \mathbf{x}_t(\theta)$ by means of the selection of Chebyshev nodes for the nodes of interpolation in (29). The spectral Legendre tau (SLT) method employs Legendre polynomials as base functions in (29). The tau approximation (TA) uses (29) to get (28) in the form of (13) where a time-dependent matrix $\mathbf{G}(t)$ appears instead of $\mathbf{A}_N$. This matrix contains $\langle \phi_j, \psi_j \rangle$ which can be approximated by using quadrature methods.

Unlike the aforementioned methods (PsC, PsT, SLT, TA), the spectral element (SE) method (called the time finite elements method as well), which was also included in the comparative study [48], approximates $\mathbb{T}(t)$ to obtain the discrete mapping. The idea lies in the splitting of the history segment $[-L, 0]$ into E number of temporary elements with the length Δt. Each element contains N_s inner points. An approximate solution is then sought in each element according to

$$\tilde{\mathbf{x}}_k(t) = \sum_{j=1}^{N_s} \phi_j(t)\,\mathbf{x}_k\left(t_{k,j}\right), t \in [-k\Delta t, -(k-1)\Delta t], \quad k = 0, 1, \ldots, E \tag{30}$$

where $t_{k,j} \in [-k\Delta t, -(k-1)\Delta t]$ are inner time instants and ϕ_j represent trial test functions. Then (12) is used to compute the system spectrum. Note that in [48], ϕ_j are Lagrange base polynomials, and the abovementioned principles were compared inter alia by using the loci of the rightmost poles here.

Vyasarayani *et al.* [83] compared the spectral tau (ST) and the spectral least-square (SLS) methods. The former one computes $\mathbf{N}, \mathbf{M}$ in (28) simply using $\mathbf{N} = \int_{-L}^{0} \boldsymbol{\varphi}(\theta)\,\boldsymbol{\varphi}^T(\theta)\mathrm{d}\theta, \mathbf{M} = \int_{-L}^{0} \boldsymbol{\varphi}(\theta)\,\varphi'^T(\theta)\mathrm{d}\theta$ where $\boldsymbol{\varphi} = \left[\phi_1, \phi_2, \ldots \phi_{N_a}\right]^T$; whereas, the latter one attempts to solve the following constrained optimization problem

$$\begin{aligned} &\min\ 0.5_{\dot{\mathbf{a}}(t)} \int_{-L}^{0} \mathbf{r}_t^2(\theta)\mathrm{d}\theta \\ &\text{s.t. } F_l\left(\boldsymbol{\varphi}^T, 0\right)\dot{\mathbf{a}}(t) = F_r\left(\boldsymbol{\varphi}^T, 0\right)\mathbf{a}(t). \end{aligned} \tag{31}$$

Base functions ϕ_j were considered as mixed Fourier basis, shifted Legendre polynomials and shifted Chebyshev polynomials, in [83]. The authors compared the methods via pole loci and stated that the ST method is easy to code and understand, and performs better than the SLS method.

To overcome computational burdens of the eigenvalues associated with the sparse $\mathbf{A}_N$, an iterative PsC method for RTDSs was presented by Ye *et al.* [84]. The sparsity of $\mathbf{A}_N$ is explored by reformulating its blocks into Kronecker products as follows

$$\mathbf{A}_N = \begin{bmatrix} \mathbf{R}_N \\ \mathbf{M}_N \otimes \mathbf{I}_n \end{bmatrix} \tag{32}$$

where $\mathbf{R}_N = \sum_{i=0}^{n_A} \mathbf{L}_i \otimes \mathbf{A}_i$ with $\mathbf{L}_i^T \in \mathbb{R}^{N+1}$ being constant Lagrange vectors. Then, the shift operation $\tilde{s} = s - s_{sh}, s_{sh} > 0$, is utilized to get eigenvalues of matrix $\tilde{\mathbf{A}}_N$ with the largest modulus, followed by the computation of $\left(\tilde{\mathbf{A}}_N\right)^{-1}$. This shift-invert preconditioning transformation technique is used for sparse eigenvalue computations with a reduced dimension by means of the implicitly restarted Arnoldi algorithm (IRA) [85] to get Krylov sequences $\{\mathbf{q}_k\}$ via $\mathbf{q}_{k+1} = (\mathbf{A}_N)^{-1}\mathbf{q}_k$. A different technique was presented by Ye *et al.* [86] where $\left(\tilde{\mathbf{A}}_N\right)^{-1} = (\Gamma_N)^{-1}\boldsymbol{\Pi}_N$, in which $\boldsymbol{\Pi}_N$ is a highly sparse companion-type constant matrix and $\tilde{\Gamma}_N = \mathbf{e}_1^T \otimes \mathbf{I}_n - \sum_{i=0}^{n_A} \mathbf{L}_i \otimes \mathbf{A}_i$. In a similar manner, a pseudospectral discretization of $\mathbb{T}(t)$ was published in [87]. The authors utilize a technique introduced in [88]; however, matrices forming $\mathbf{T}.(\Delta t)$ are reformulated by using Kronecker products to reduce their dimensions. Then, the rotation-and-amplification operation $\tilde{s} = \alpha\exp(-\theta \mathrm{j})\,s, \alpha > 1, \theta > 0$, is made prior to the generation of Krylov vectors via the IRA, to accelerate its convergence rate. This operation implies that system poles are first rotated by θ and then amplified by α.

Fabiano [89] suggested the approximation of A (therein called a semidiscrete approximation) for a NTDS via linear spline functions where he proved Trotter-Kato type semigroup convergence [63] for this scheme as well. The same author used this scheme to investigate the DIS problem in [90]. In both papers, the exact eigenvalues of $\mathbf{A}_N$ were used to measure the accuracy of the approximation.

Note that a summary of pseudospectral discretization methods according to (9)-(14), (29) can be found e.g. in the book by Breda *et al.* [91], and these methods were sufficiently applied to biological models (see e.g. [92]). The extension to inter alia systems with uncertain parameters was presented in [93].

B. SEMIDISCETIZATION AND FULL-DISCRETIZATION METHODS

Although semi-discretization (SD) methods and full-discretization (FD) methods, as well as integral and quadrature ones, in fact, provide the discrete approximation of $\mathcal{T}(t)$, they are considered separately herein. Note, however, that these methods are primarily applicable to time-periodic systems. Let us briefly express the idea of SD methods for the following particular RTDSs (23) [94]: First, L is divided into l intervals with the length Δt, i.e. $L = l\Delta t$. Then, the solution of the RTDS (23) on each time interval $t \in [k\Delta t, (k+1)\Delta t]$ reads

$$\begin{aligned}\mathbf{x}(t) &= \exp\left(\mathbf{A}_0 (t - k\Delta t)\right)\mathbf{x}(k\Delta t) \\ &\quad + \int_{k\Delta t}^{t} \exp\left(\mathbf{A}_0 (t-\xi)\right)\mathbf{A}_1 \mathbf{x}(\xi - L)\,\mathrm{d}\xi. \end{aligned} \tag{33}$$

The key point lies in the approximation of the delay term as for $\mathbf{x}(t - l\Delta t) \approx \sum_{i=0}^{i_{\max}} d_i(t)\,\mathbf{x}_{k+i-l}$ where $\mathbf{x}_{k+i-l} = \mathbf{x}((k+i-l)\Delta t)$, $i_{\max} \leq l$, and $d_i(t)$ are weight functions given by the particular interpolation method and its order, see [94] for further details. If it is set $t = (k+1)\Delta t$, (33) yields

$$\begin{aligned}\mathbf{x}_{k+1} &= \tilde{\mathbf{A}}_0 \mathbf{x}_k + \sum\nolimits_{i=0}^{i_{\max}} \tilde{\mathbf{A}}_{1,i}\mathbf{x}_{k+i-l}, \\ \tilde{\mathbf{A}}_0 &= \exp\left(\mathbf{A}_0 \Delta t\right), \\ \tilde{\mathbf{A}}_{1,i} &= \int_0^{\Delta t} \exp\left(\mathbf{A}_0(\Delta t - t)\right)\mathbf{A}_1 d_i(t)\,\mathrm{d}t. \end{aligned} \tag{34}$$

In contrast to SD methods, the family of FD methods is based on discretizing both the state term and the time-delay term [95]. Equation (33) can be reformulated so that it includes $\mathbf{x}((k+1)\Delta t - \xi - L)$ rather than $\mathbf{x}(\xi - L)$; then, the $j_{\max}$ th order interpolation is $\mathbf{x}((k+1)\Delta t - \xi - L) \approx \sum_{j=0}^{j_{\max}} d_j(\xi)\,\mathbf{x}_{k+1-j-l}$.

Tweten *et al.* [96] provided the comparison of SD, SE and SLT methods applied to autonomous (time-invariant) and time-periodic RTDSs via inter alia the distance of the rightmost poles of the system from their approximations. The authors observed that the SE method had the best convergence rate and the SLT yielded the shortest computation time, while the SD method fell behind in both the performance measures.

Lehotzky and Insperger [97] utilized SD for the stability analysis of digitally controlled RTDS. An improved FD method with Lagrange polynomial interpolation to predict milling stability was developed by Tang *et al.* [98] where the authors also compared this technique with SD and numerical integration (NI) methods. Again, the distance of critical poles served as the benchmark tool.

C. NUMERICAL INTEGRATION AND DIFFERENTIAL QUADRATURE METHODS

Roughly speaking, NI methods are based on a numerical approximation of the right-hand side of the FDE solution – e.g. as in (33) - via different rules [99]. If $L/\Delta t \notin \mathbb{N}$, the appropriate interpolation to approximate the delayed term is used. For instance, the linear interpolation yields

$$\begin{aligned}\mathbf{x}(k\Delta t - L) &= \mathbf{x}(k\Delta t - q\Delta t - r) \\ &\approx r/\Delta t\mathbf{x}((k-1)\Delta t - q\Delta t) \\ &\quad + (\Delta t - r)/\Delta t\mathbf{x}(k\Delta t - q\Delta t) \\ &= r/\Delta t\mathbf{x}_{k-1-q} + (\Delta t - r)/\Delta t\mathbf{x}_{k-q} \end{aligned} \tag{35}$$

where $L = q\Delta t + r$, $q \in \mathbb{N}$, $r \in [0, \Delta t)$.

Unfortunately, these methods are used mostly to solve time-variant (periodic) TDS engineering problems, see e.g. [100]. Zhang *et al.* [101] improved the NI method by using the Lagrange form interpolating polynomial to approximate the delayed terms and construct a discrete dynamical map for the damped Mathieu equation (in time-invariant and time-periodic form) with time delays, where the obtained approximation is in the form (14). A comparison with the SD method was also provided resulting in the observation that updated NI method has a faster computational speed than the SD method.

The key step of differential quadrature (DQ) methods lies in that a partial derivative of a function with respect to a coordinate direction (at a sampling grid point within an interval along that direction) is approximated as a linear weighted sum of function values at the sampling grid points within the whole interval. Hence, the weighting coefficients for discretization of the derivative are to be determined. Then, the FDE can be discretized as a series of algebraic equations, see e.g. [102] for a time-periodic system. Dong *et al.* [103] developed a computationally efficient stability analysis method for NTDSs.

D. CONTOUR INTEGRAL METHOD

A general algorithm for computing system poles inside a defined open disk in $\mathbb{C}^+$ was proposed by Chen and Liu [104], and Chen and Dai [105]. It was utilized to investigate the local asymptotic stability of the positive equilibrium for the n-dimensional Lotka-Volterra system and to compute the rightmost characteristic roots (poles), respectively. The main idea of the algorithm is based on the transform of the nonlinear eigenvalue problem $\mathbf{\Phi}(s_k)\,\mathbf{x} = 0$ to the general eigenvalue problem $\mathbf{H}_{m1}\mathbf{x} = s_k\mathbf{H}_{m2}\mathbf{x}$ where Φ is the characteristic matrix, i.e., $\Delta(s) = \det\mathbf{\Phi}(s)$, $s_k \in \mathbb{C}$, and $\mathbf{H}_{m1}$, $\mathbf{H}_{m2}$ are Henkel matrices formed by using complex moments approximated by the trapezoidal rule. The Mikhailov stability criterion has to be included in this framework as well.

Xu and Wang [106] and Xu *et al.* [107] proposed a numerical scheme for calculating the rightmost characteristic roots of a given NTDS as well as the characteristic roots other than the rightmost ones based on the proved Mikhailov stability criterion and its equivalent integral form. Moreover, the characteristic root loci of NTDSs compared to RTDSs was briefly discussed in the latter publication. The crucial result is the following.

Theorem 2: Define

$$F(\Omega, a) := \int_0^{\Omega} \mathrm{Re}\left(\frac{\Delta'(a + \mathrm{j}\omega)}{\Delta(a + \mathrm{j}\omega)}\right) \mathrm{d}\omega,$$

and assume that the characteristic function $\Delta(s)$ has no roots on $\mathbb{C}^0$ and it holds that

$$\sup_{\mathrm{Re}s>0, |s|\to\infty} \left|d_{n_\Delta}(s)\right| < 1. \tag{36}$$

Then a NTDS is asymptotically stable if and only if for sufficiently large $\Omega > 0$, one has

$$\mathcal{N} = \mathrm{round}\left(\frac{n}{2} - \frac{F(\Omega, 0)}{\pi}\right) = 0 \tag{37}$$

where $\mathcal{N}$ means the number of poles in $\mathbb{C}^+$.

The spectral abscissa α can then be estimated based on Theorem 2 as follows. Assume that there is a real number a_1 such that for sufficiently large $\Omega > 0$ it holds that $\mathcal{N} = \mathrm{round}(0.5n - F(\Omega, a_1)/\pi) > 0$, i.e. $\alpha > a_1$, and let there exist a_2 such that one has $\mathcal{N} = \mathrm{round}(0.5n - F(\Omega, a_2)/\pi) = 0$, hence $\alpha < a_2$. If $a_2 - a_1$ is small enough, the abscissa can be estimated as $\alpha \approx 0.5(a_1 + a_2)$.

E. LAMBERT W FUNCTION

Duan *et al.* [108] provided the calculation of the decay rate, i.e. $K\exp(\alpha(\cdot)t), K \in \mathbb{R}$, via the Lambert W function. Yi *et al.* [109] presented *LambertW_DDE* Matlab toolbox implementing the Lambert W function approach for the analysis and control of TDSs in terms of stability, observability, controllability, and observer and controller design via eigenvalue assignment within this framework.

Cepeda-Gomez and Michiels [110] proved for a particular second order system that there is no one-to-one correspondence between the branches of the characteristic roots associated with the system but only two branches suffice to find the complete spectrum of the system, namely $k = -1$, $k = 0$. Moreover, the principal branch ($k = 0$) can be used not only for the dominant root, but also for some non-dominant roots. An extension of this method to the nth order system was given by Choudhary *et al.* [111] who confirmed that the whole eigenspectrum can be associated with only two real branches of the Lambert W function. The applicability of the method was also improved by the introduction of a new class of TDSs and the corresponding transformation into the proposed common canonical form. In his technical note [112], Cepeda-Gomez proved by an example that all the characteristic roots of system (23) can be found using the non-principal branch $k = -1$.

Surya *et al.* [113] developed a homotopy continuation method to find the characteristic roots of RTDSs with multiple delays. A homotopy parameter μ was introduced into the characteristic equation so that this equation contains only one exponential term, corresponding to the largest delay, and all the characteristic roots can be expressed in terms of the Lambert W function for $\mu = 0$. The original characteristic equation was recovered for $\mu = 1$. Then a pseudo-arclength continuation was used to trace the roots as a function of μ.

F. SPECIAL NUMERICAL, SEMI-ANALYTIC AND ANALYTIC METHODS

This part of the subsection covers various methods which have contributed to the knowledge of the TDS spectrum and cannot be directly assigned to any class mentioned above. Boussaada *et al.* [114] extended the spectral projection methodology [115] for delay differential-algebraic systems (that can be characterized by a RTDS model with mixed dimensions or a special possibly singular NTDS) by introducing an appropriate bilinear form associated with the special RTDS model. Then, a procedure scheme for computing associated spectral projection is described. The question whether the solution of the model can be represented by a series of elementary solutions is simultaneously addressed. The proposed direct method provides the central manifold approximation for lossless propagation model without the use of the central manifold theorem and the structure reconstruction. Conditions for the convergence of the power series are characterized by the system poles and associated spectral projection. Specific features of $\mathcal{T}(t)$ and $\mathcal{A}$ were also utilized. Unfortunately, the method is highly mathematically involved and hard to understand.

Breda [116] presented the purely analytic study of the characteristic roots of the scalar RTDS with one delay with either real or complex coefficients. The focus was on the robust analysis of the pole loci in the complex plane with respect to the variation of the coefficients. Relevant stability charts and boundaries were eventually obtained.

Some very interesting results were derived by Bonnet *et al.* [51] where loci of NTDSs (with commensurate delays) poles asymptotic to $\mathbb{C}^0$ (not necessarily the rightmost ones) were calculated analytically by means of an asymptotic approximation up to the selected order. They proved that although the asymptotes are given solely by Σ_{ess}, the number, shape and type of pole chains depend on other parameters of $\Delta(s)$. The paper also addressed the H_∞ stability where necessary and sufficient conditions were derived. These results were then extended in [117] and especially in [118] where some classes of systems with multiple chains of poles asymptotic to a same set of points on $\mathbb{C}^0$ were addressed, i.e. poles with high moduli were determined. In the time domain, neutral systems with poles approaching $\mathbb{C}^0$ were studied in [119] as well. The findings derived in [51] were then used in YALTA Matlab toolbox [120] dedicated to the H_∞ stability analysis of both RTDSs and NTDSs with commensurate delays given by their transfer functions, based on pole loci. Poles with small moduli were approximated by using a finite-dimensional (Padé-2) approximation.

Another Matlab toolbox, the QuasiPolynomial mapping Rootfinder (QPmR), was enhanced by its original authors in [52] and called advanced QPmR (aQPmR). This algorithm is generally based on the numerical searching of intersections of the curves satisfying $\mathrm{Re}\,(\Delta\,(\beta + \mathrm{j}\omega, \cdot)) = 0$ and $\mathrm{Im}\,(\Delta\,(\beta + \mathrm{j}\omega, \cdot)) = 0$ inside a region of interest covered by a rectangular mesh grid: $(\beta_0, \beta_1, \ldots \beta_{k_{\max}}) \times \mathrm{j}\,(\omega_0, \omega_1, \ldots \omega_{l_{\max}})$. The main improvement was given by the inclusion of recursive grid density adaptation while the use of the Symbolic Math Toolbox was avoided in this version – this step resulted in at least twice as fast the computing rates.

Jarlebring *et al.* [121] applied the Arnoldi method, which is well-established to finite-dimensional systems, to TDS eigenvalue problems. The adaption was based on the formulation of a more general problem as an eigenvalue problem associated with an operator and only finite-dimensional operations with matrices could be implemented. The Fourier cosine transform was used here to deal with distributed delays.

In his brief paper, Bortz [122] studied the pole loci of a RTDS with two delays by means of a special series expansion.

Traditional methods based on the substitution of exponential terms still do not stand aside; to name just a result, Niu *et al.* [123] utilized a Padé-approximation based method to investigate the spectrum of a RTDS. This approximation generally yields a satisfactory phase approximation, but introduces a non-minimum phase artifact in the initial transient response.

Although the Hermite-Biehler theorem is usually used to test exponential or parameter-dependent stability, the recent work of Wang *et al.* [125] extended it to reveal the information about the characteristic roots distribution of a RTDS and even a NTDS with commensurate delays. The main result was given as follows.

Theorem 3: Let $\Delta^*(s)$ be a quasipolynomial (retarded or neutral one) with $n = \deg_s(\Delta^*(s))$, the degree of commensuracy equal to n_C and no roots on $\mathbb{C}^0$. Then $\Delta^*(s)$ possesses $\mathcal{N}$ roots in $\mathbb{C}^+$ if and only if

$$\gamma_I\left(\Delta^*\right) = \begin{cases} 4nl + n_C + 1 - 2\mathcal{N} & \text{for } n_C \text{ even} \\ 4nl + n_C - 2\mathcal{N} & \text{for } n_C \text{ odd} \end{cases} \tag{38}$$

or

$$\gamma_R\left(\Delta^*\right) = \begin{cases} 4nl + n_C - 2\mathcal{N} & \text{for } n_C \text{ even} \\ 4nl + n_C + 1 - 2\mathcal{N} & \text{for } n_C \text{ odd} \end{cases} \tag{39}$$

for a sufficiently large integer l where

$$\begin{aligned} \gamma_I\left(\Delta^*\right) &= (-1)^{Q-1}\operatorname{sgn}\left(\Delta_I^*\left(\omega_{I,Q-1}^+\right)\right) \\ &\quad\cdot \begin{pmatrix} \operatorname{sgn}\left(\Delta_R^*\left(\omega_{I,0}\right)\right) - 2\operatorname{sgn}\left(\Delta_R^*\left(\omega_{I,1}\right)\right) \\ + \cdots + (-1)^{Q-1}\, 2\operatorname{sgn}\left(\Delta_R^*\left(\omega_{I,Q-1}\right)\right) \end{pmatrix}, \\ \gamma_R\left(\Delta^*\right) &= (-1)^{P}\operatorname{sgn}\left(\Delta_R^*\left(\omega_{R,P}^+\right)\right) \\ &\quad\cdot \begin{pmatrix} 2\operatorname{sgn}\left(\Delta_I^*\left(\omega_{R,1}\right)\right) - 2\operatorname{sgn}\left(\Delta_I^*\left(\omega_{R,2}\right)\right) \\ + \cdots + (-1)^{P-1}\, 2\operatorname{sgn}\left(\Delta_I^*\left(\omega_{R,P}\right)\right) \end{pmatrix}, \end{aligned}$$

in which $0 = \omega_{I,0} < \omega_{I,1} < \cdots < \omega_{I,Q-1}$ are real distinct roots of $\Delta_I^*(\omega)$ with odd multiplicities in $[0, \Omega_I)$, $\Omega_I = 2\pi l + \pi/(2n)$ for n_C even or $\Omega_I = 2\pi l$ for n_C odd, $\operatorname{sgn}\left(\Delta_I^*\left(\omega_{I,Q-1}^+\right)\right)$ denotes the sign of $\Delta_I^*(\omega)$ soon after the occurrence of the zero $\operatorname{sgn}\left(\Delta_I^*\left(\omega_{I,Q-1}\right)\right)$, and $0 = \omega_{R,0} < \omega_{R,1} < \cdots < \omega_{R,P}$ are real distinct roots of $\Delta_R^*(\omega)$ with odd multiplicities in $[0, \Omega_R)$, $\Omega_R = 2\pi l$ for n_C even or $\Omega_R = 2\pi l + \pi/(2n)$ for n_C odd, $\operatorname{sgn}\left(\Delta_R^*\left(\omega_{I,P}^+\right)\right)$ denotes the sign of $\Delta_R^*(\omega)$ soon after the occurrence of the zero $\operatorname{sgn}\left(\Delta_R^*\left(\omega_{I,P}\right)\right)$.

Model order reduction and finite dimensional approximation methods which can be used to estimate the system spectrum or those based on it should be concisely outlined as well. To introduce just a few, Saadvandi *et al.* [126] proposed a new technique to approximate a second order TDS in the general concept of the dominant pole algorithm, which was based on the residual expansion of the system transfer function in the form $G(s) = \mathbf{d}^T\Delta^{-1}(s)\,\mathbf{f}$, $\mathbf{d}, \mathbf{f} \in \mathbb{R}^n$, as

$$\begin{aligned} G(s) &= \sum\nolimits_{k=1}^{\infty} \frac{\mathbb{R}_k}{s - s_k}, \\ R_k &= \frac{\mathbf{d}^T\mathbf{x}\mathbf{y}^T\mathbf{f}}{\mathbf{y}^T\Delta'(s_k)\,\mathbf{x}}, \quad \Delta'(s) = \frac{\mathrm{d}\Delta(s)}{\mathrm{d}s}, \end{aligned} \tag{40}$$

where R_k is the residue, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are right and left eigenvectors, respectively, corresponding to the pole s_k. In order to avoid the scenario that the algorithm converges to the same pole several times, the so-called deflation is usually made which cannot be applied to linear systems in some cases. Hence, the main contribution of this work was in the proposition of an alternative technique that essentially removes the computed poles from the system's input and output vectors. Note that these authors computed the pole dominancy as

$$\rho_k = \frac{|R_k|}{|\mathrm{Re}\,(s_k)|}. \tag{41}$$

The algorithm was further extended to parametric systems in [127]. Two approaches to find dominant poles were presented, one is based on a one-by-one procedure and the other is in an independent manner [128]. Both these methods are built on the Ritz values to get the so-called Hermite interpolation.

Ionescu and Iftime [129] presented a moment matching approach for infinite-dimensional systems based on the unique solution of an operator Sylvester equation. The solution preserves poles as well as zeros of the original model.

Theory, algorithms and software to approximate the finite-impulse response (FIR) filters by stable LTI systems were proposed by Michiels and Ünal in [130]. In fact, ann FIR filter can be described by the following input-output map

$$\begin{aligned} y(t) &= \int_{-L}^{0} \mathbf{C}\exp(-\mathbf{A}\tau)\,\mathbf{B}u(t+\tau)\,\mathrm{d}\tau, \\ &\qquad \mathbf{A} \in \mathbb{R}^{n\times n}, \quad \mathbf{B} \in \mathbb{R}^{m\times n}, \ \mathbf{C} \in \mathbb{R}^{l\times n}, \end{aligned} \tag{42}$$

which is equivalent to the transfer function

$$\mathbf{G}(s) = \mathbf{C}\,(s\mathbf{I} - \mathbf{A})^{-1}\,(\mathbf{I} - \exp(-L\,(s\mathbf{I} - \mathbf{A})))\,\mathbf{B}. \tag{43}$$

The approach is based on the theory of functions of matrices, namely the Padé approximation and the Kronecker product were utilized in [130]. In the approach, pole-zero cancelations are explicitly taken into account, and the approximation performance is measured inter alia by the spectral abscissa value.

Last but not least, Pontes Duff *et al.* [131] utilized the Lambert W function to solve the following SISO optimization problem: Given a stable model $G(s) \in H_2$ and $\tau > 0$ find a model

$$H_m(s) = \frac{b_m}{s - a_m \exp(-\tau s)} \in H_2, \quad a_m, b_m \in \mathbb{R} \tag{44}$$

so that

$$\min_{a_m, b_m} \|G(s) - H_m(s)\|_2 . \tag{45}$$

It holds that the optimality conditions can be given by using equalities expressed by means of $H_m(s_k)$, $H_m'(s_k)\, G(s_k)$ and $G'(s_k)$ where s_k is a pole of $H_m(s)$ (from the infinite set) and the apostrophe $\cdot'(s)$ stands for the derivative with respect to s.

As the last note to this subsection, let us briefly introduce some recent basic results on the spectrum of exponential polynomials that can be used for computing Σ_{ess} (see (4) and Proposition 2). Some decent general findings on Σ_{ess} can be deduced from the aforementioned works [51], [117], [118]. Corless [132] proved that pseudospectra of matrix polynomials expressed in other bases are unaffected by drawing the matrix coefficients from certain structured families; however, the author also showed that this behavior is not universal. Sepulcre [133] gave a complete description of the set (defined as the closure set of the real projections of zeros of an exponential polynomial $p(s) = \sum_{i=1}^{n} p_i \exp(\omega_i s)$ with real frequencies ω_i linearly independent over the rationals) and proved that it is invariant with respect to the moduli of p_i. The reverse of this result of invariance was analyzed as well. The proof was given in [134] that the real projection of each zero of any function $p(s)$ in a large class of exponential polynomials is an interior point of the closure of the set of the real parts of the zeros of $p(s)$.

Selected results from this section are summarized in Table 1 where the specifically studied TDS model, used methodology (an approximation technique) and the problem type are introduced.

VI. STABILITY STUDIES RELATED TO THE TDS SPECTRUM

This section contains methods concerning pole-loci-related stability issues, i.e. the behavior of the rightmost imaginary poles and those lying close to $\mathbb{C}^0$. The latter group is then divided into several parts, in which exponential, BIBO, H_∞, DDS and DIS stability are considered separately. In some cases, fixed yet uncertain delays or parameters are considered, and in others stability under small delay variations is studied.

Several abovementioned types of stability are considered with the following two scenarios – first, all the parameters and delays are fixed with known values; second, some parameters and/or delays are unknown but within a particular interval or a region (see DDS and DIS in Definition 3) and, hence, the task is to find the stable interval(s) or region(s).

A. EXPONENTIAL AND ASYMPTOTIC STABILITY

Recall that time-domain methods estimate the (exponential) decay rate (or, also called the stability degree) only (see e.g. [135], [136]) or they just prove the existence of imaginary roots [137]; thus, they are mostly not covered in this survey. Indeed, Milano [138] stated that finding the Lyapunov function which implies finding a solution of an LMI problem does not solve the eigenvalue problem (analysis and/or synthesis). Moreover, the conditions of the Lyapunov–Krasovskii stability theorem and the Razumikhin theorem are only sufficient and cannot be used to find the delay stability margin. Within this framework, weighted integral inequalities, like the Jensen and the Wirtinger inequality, are very favorite tools to investigate asymptotic [139] or exponential stability [140].

Damak *et al.* [141], in their pioneer work, presented the idea of a bridge between Lyapunov–Krasovskii approaches and spectral ones for systems governed by linear difference equations with commensurate delays, by means of the analytic solution, yielding necessary and sufficient asymptotic stability conditions. Zhang and Sun [142] studied the stability of two benchmark systems by using SD, PsC and Lyapunov stability theory. It was found that the Lyapunov method is usually conservative with the exception of the complete Lyapunov functional due to Gu [24], [143], which gives highly accurate predictions with little conservatism.

In order to recall some results introduced in the preceding section, it is worth referring to the work of Tweten *et al.* [96] where asymptotic stability was studied by the comparison of SD, SE and SLT methods. The decision about the local asymptotic stability of the positive equilibrium of the Lotka-Volterra system by means of the contour integral method was the primary goal of [104]. Exponential stability analysis of RTDSs subjected to a digital controller using the SE method was performed by Lehotzky and Insperger [97]. The same type of stability was further the aim of papers by Zhang *et al.* [101] (NI and SD methods) and Dong *et al.* [103] (the DQ method).

The PsC method was employed by Milano and Anghel [144] to compute the eigenvalues of delayed cyber-physical power systems (DCPPS) with single time delay, so that their impacts on system small signal stability were evaluated. In [138], the method was further compared with linear multi-step and Runge–Kutta discretization scheme of $\mathcal{T}$ in computing the rightmost poles of large DCPPS with multiple delays. Numerical studies revealed that the PsC method is more accurate compared to the others and with less computational burden.

Domoshnitsky *et al.* [145] investigated the exponential stability of the scalar undamped second order RTDS via the so-called W-method that is based on a transformation of the given differential equation to an operator equation by the

TABLE 1. Results on pole loci calculation, computation and approximation (Section v).

Authors	Model	Methodology/technique	Problem type
Lehotzky and Insperger [48]	(d)RTDS, multiple delays	PsT, PsC, SLT, SE, TA (comparison),	Approximation of A and $\mathsf{T}(t)$, computing rightmost poles
Wu and Michiels [60]	RTDS, multiple commensurate and incommensurate delays	PsC	Approximation of A, computing poles in $\mathbb{C}^+$
Vyasarayani *et al.* [83]	Scalar RTDS, single delay	ST, SLS (comparison)	Computing all poles
Ye *et al.* [84], [86]	RTDS (DCPPS)	PsC, Kronecker product, shift operation, IRA	Approximation of A, computing rightmost low frequency (critical) poles
Ye *et al.* [87]	RTDS (DCPPS)	PsC, Kronecker product, shift-and-amplification operation, IRA	Approximation of $\mathsf{T}(t)$, computing rightmost low-frequency poles
Breda *et al.* [93]	RTDS with uncertain parameters	PsC	Approximation of $\mathsf{T}(t)$, computing all poles
Insperger and Stépán [94]	Autonomous and time-periodic TDS	SD	Finite-dimensional approximation and eigenvalue computation
Tweten *et al.* [96]	Autonomous and time-periodic RTDS	SD, SE, SLT (comparison)	Finite-dimensional approximation
Tang *et al.* [98]	RTDS, single delay	FD, SD, NI (comparison)	Finite-dimensional approximation, milling stability prediction
Chen and Dai [105]	RTDS	Contour integral method	Computing the rightmost poles
Xu and Wang [106], [107]	NTDS	Mikhailov criterion (contour integral method)	Computing the rightmost poles (and also some others) and the spectral abscissa
Duan *et al.* [108]	RTDS, single delay	Lambert W function	Computing the rightmost poles (decay rate)
Yi *et al.* [109]	RTDS, single delay	Lambert W function	Computing all poles, stability, observability, controllability, controller design
Cepeda-Gomez and Michiels [110]	RTDS, second order, single delay	Lambert W function	Computing the rightmost poles and the complete spectrum
Choudhary *et al.* [111]	RTDS, single delay	Lambert W function	Computing the complete spectrum
Surya *et al.* [113]	RTDS, multiple delays	Homotopy (pseudo-arclength) continuation method, Lambert W function	Computing the rightmost poles and the complete spectrum
Boussaada *et al.* [114]	Delay differential-algebraic system (singular NTDS)	Spectral projection, bilinear matrix form	Finite-dimensional approximation and eigenvalue computation
Breda [116]	Scalar RTDS, single delay, complex coefficients	Analytic method	Computing the complete spectrum, asymptotic stability
Bonnet *et al.* [51]	NTDS, commensurate delays	Semi-analytic method	Computing poles asymptotic to $\mathbb{C}^+$
Nguyen *et al.* [117], [118]	NTDS, commensurate delays	Semi-analytic method	Computing multiple chains of poles asymptotic to the same point on $\mathbb{C}^+$
Avanessoff et al. [120]	RTDS, NTDS, commensurate delays	Semi-analytic method, Padé-2 approximation	Computing the complete spectrum including high-modulus poles
Vyhlídal and Zítek [52]	RTDS, NTDS, multiple commensurate and incommensurate delays	aQPmR	Computing spectrum inside the defined region
Jarlebring *et al.* [121]	(d)RTDS, single delay	Arnoldi method, Fourier cosine transform	Computing the complete spectrum
Bortz [122]	Scalar RTDS, two delays	Series expansion	Computing the complete spectrum
Niu *et al.* [123]	RTDS	Padé approximation	Computing the complete spectrum
Wang *et al.* [125]	RTDS, NTDS	Hermite-Biehler theorem	Computing the number of poles in $\mathbb{C}^+$

following substitution

$$x(t) = \int_0^t W(t, s) z(s) \, \mathrm{d}s \tag{46}$$

where $W(t, s)$ is the Cauchy function for some known exponentially stable equation. The paper, in some sense, dealt with the DDS problem since the authors showed that although the delay-free system can be unstable, the delayed-one can be exponentially stable.

The problem of a stability test of NTDSs via the Mikhailov criterion was studied in [106]. The criterion was characterized by an auxiliary function associated with the characteristic quasipolynomial, not by the characteristic function itself. The authors used the criterion to compute the rightmost spectrum and to derive a graphical exponential stability criterion based on the knowledge whether the Nyquist plot encircles the origin of the complex plane or not.

A very delicate critical case when there is a sequence of the rightmost poles with real parts converging to zero for mixed RTDSs and NTDSs (modeled by operator differential equations in a Hilbert space) was studied in [119]. The mixed structure was given by $\mathbf{H} \neq \mathbf{0}$ but with $\det \mathbf{H} = 0$. In this case, the system cannot be exponentially stable; hence asymptotic and strong stability were analyzed by means of a Riesz basis of invariant finite-dimensional subspaces and the boundedness of the resolvent in some subspaces of a special decomposition of the state space.

A lot of results on the behavior of imaginary poles have been obtained via several methodologies, especially for poles with the multiplicity higher than one. It is clear that these findings are decisive for exponential and asymptotic stability (see Propositions 4 and 5) and also for the DDS problem. A criterion for the rightmost poles lying exactly on $\mathbb{C}^0$ (including the origin) obtained by means of the extended Hermite-Biehler theorem for both RTDSs and NTDSs was presented by Wang *et al.* [146].

Boussaada and Niculescu [147] studied the multiple zero singularity, namely the case when the algebraic multiplicity is two and the geometric one equals one (i.e., the so-called Bogdanov-Takens singularity) by means the functional confluent Vandermonde matrix as well as some classes of the functional Birkhoff matrix. An explicit recursive formula for the so-called LU-factorization was proposed as well. It was shown that the admissible multiplicity of the zero spectral value is constrained by the Pólya and Szegö bound (n_{PSB}) [148] that arises from the principle argument and the bound equals the degree of the corresponding quasipolynomial. These results were then improved in [149] where it was shown that a given imaginary multiple pole with a non-vanishing frequency never reaches n_{PSB}, and a bound more precise than the n_{PSB} generic bound was established. However, an example of a scalar RTDS with two delays demonstrated that the multiplicity of real spectral values might reach the n_{PSB}. The corresponding system is asymptotically stable and its spectral abscissa corresponds to this maximal allowable multiple root located on $\mathbb{C}^0$.

Louisell [150] presented an approach to determining the imaginary axis eigenvalues of a matrix delay equation. With a full rank delay coefficient matrix, the approach requires the computation of the generalized eigenvalues of a pair of matrices which are a quarter of the size used in currently known matrix-based or operator approaches. The frequency sweeping methodology has been intensively used to study the (multiple) imaginary pole loci.

To name just a few important results, it is worth referring to works of Li *et al.* [151], [152] in the first place, where the authors tackled the simple and multiple imaginary roots, respectively. However, the former paper did not produce considerably new results, the latter one used the Puiseux series expansion given by

$$\Delta s_k = \sum\nolimits_{i=1}^{n} c_i \left(\Delta\tau\right)^{\frac{i}{m}} \tag{47}$$

to investigate the poles behavior, where s_k is any m-multiple imaginary pole and c_i are complex coefficients. The authors inter alia proved the result known for a simple root that whenever an imaginary multiple root appears as $\boldsymbol{\tau}$ increases, the number change of the unstable roots is the same (i.e. the root invariance property). Since stability regions can be computed using this methodology, the DDS problem can also be solved. These results were then more deeply formalized and extended in [153] and [154], where inter alia the so-called dual Puiseux series $\Delta\tau\ (\Delta s_k)$ was defined and further utilized.

The Puiseux series played a crucial role to generate other important results as well. Cai *et al.* [155] solved the same problem as introduced above: First, the Weierstrass preparation theorem was used to get an explicit expression of the coefficients of the algebraic equation equivalent to the characteristic quasipolynomial in infinite power series of delay parameter; the determinations of such power series coefficients are related to the computation of residues of meromorphic functions. Second, the classic Puiseux-Newton diagram algorithm was used to calculate the algebraic expansions of the reduced equation directly. As a result, the asymptotic behavior of root loci near singular points of the quasipolynomial equation was obtained.

Méndez-Barrios *et al.* [156] investigated the behavior of a multiple characteristic root and the corresponding stability issue under small variations of the delay parameter as well. The authors first utilized the Weierstrass preparation theorem to construct the Weierstrass polynomial that captures all the stability information corresponding to the case of the multiple critical pole. Then, the so-called pseudopolynomial to construct the Newton diagram was introduced, and consequently, the diagram was applied to compute the corresponding Puiseux series and the crossing directions of the critical pole to get the asymptotic behavior of the critical poles when the delay varies. The reader are referred to the cited paper for more detail.

Bouzidi *et al.* [157] presented a twofold result: First, a new approach for the computation of the critical pairs was presented. Second, the variations of Δs_k with respect to $\Delta\tau$ in the neighborhood of the critical pair (s_c, τ_c) were computed. The former one was achieved by using the Rekasius or Möbius transformation that reduces the computation of the critical pairs of a quasipolynomial to that of real solutions of a zero-dimensional polynomial system in two variables, i.e., a system admitting a finite number of complex solutions. This was

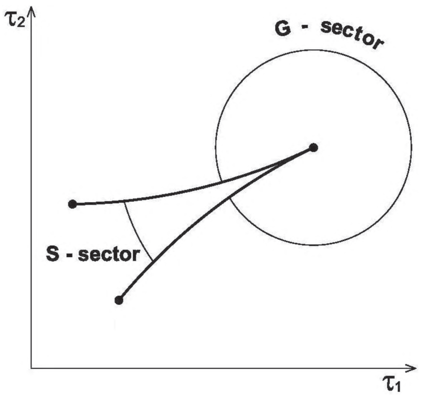

FIGURE 2. G-sector and S-sector [159]–[161].

done via advanced computer algebra technique, the so-called rational univariate representation [158], which is a one-to-one mapping between the solutions of the polynomial system and the roots of a univariate polynomial. It is worth noting that the following form of the Möbius transformation (mapping $\omega \in \mathbb{R} \cup \{\infty\}$ to the unit circle $\mathbb{T} := \{x \in \mathbb{C} : |x| = 1\}$) was used

$$\exp(-\tau j\omega) \to \frac{x - j}{x + j}. \tag{48}$$

The critical pairs can be computed by solving the following identities $\mathrm{Re}\{p(\omega, x)\} = \mathrm{Im}\{p(\omega, x)\} = 0$ where $p(\omega, x)$ is the bivariate polynomial corresponding to $\Delta(s)$. The critical delays are then obtained as follows

$$\tau_k = \omega^{-1}\left(\tan^{-1}\left(\frac{2x}{x^2 - 1}\right) + k\pi\right), \quad k \in \mathbb{Z}. \tag{49}$$

Note also that the well-known Rekasius substitution (29) is obtained by the setting $x = -T\omega, T \in \mathbb{R}$. An alternative Möbius mapping reads $\exp(-\tau j\omega) \to z = u + jv \in \mathbb{T}$. The latter sub-result of [157] included an efficient algorithm to compute the different terms in the Puiseux series.

The movement of double, triple and quadruple imaginary poles when two delays are subjected to small deviations was analytically studied without using the Puiseux series in [159]–[161], respectively. Two sectors, great (G) and small (S), were introduced in the neighborhood of the point in the delay parameter space that causes the multiple imaginary root, see Fig. 2. This critical point constitutes a cusp in the stability crossing curve. When the delay parameters move into the G-sector, one root (two roots) move(s) to $\mathbb{C}^+$, and the other one (two others) move(s) to $\mathbb{C}^-$ for the double (quadruple) imaginary pole. If the parameters move into the S-sector, then one (three) of the roots move(s) to one half-plane, and the remaining root moves to the other half-plane. For the triple pole, it was proved in the cited paper that the stability crossing curves are smooth - two roots move to one half-plane and one root goes to the other half-plane.

B. H_∞ AND BIBO STABILITY

As mentioned above, Bonnet *et al.* [51] provided a thorough H_∞ analysis by means of estimating the pole loci asymptotic to $\mathbb{C}^0$ which was then implemented by using YALTA software package by Avanessoff *et al.* [120]. H_∞ stability of some classes of TDSs with multiple chains of poles asymptotic to the same set of points on $\mathbb{C}^0$ was studied in [117] and [118]. Similar approximation tools were also utilized to analyze poles of a ''small'' modulus and the corresponding BIBO and H_∞ stability for a NTDS with a single delay in [161].

C. STRONG AND ROBUST STABILITY

Strong stability has been introduced in Definition 2. By the notion of robust stability we mean the ability of a system to remain stable (in some sense) with respect to parameters fluctuations, or under model or parameter uncertainties. A strong stability criterion for NTDSs was presented in [103], the derivation of which was made by means of the DQ method. Rabah *et al.* [119], inter alia studied conditions under which a mixed RTDS/NTDS is strongly stable.

Du *et al.* [163] presented necessary and sufficient conditions for exponential stability of TDS governed by differential-algebraic equations. In particular, the robustness of this type of stability was studied when the equation is subject to structured perturbations. A computable formula for the structured stability radius was also derived.

Otten and Mönnigmann [164] proposed an optimization method for parametrically uncertain delay differential equations with state-dependent delays. The central idea of the optimization is to stay off the stability boundaries in the parameter space. As a result, dynamical properties such as stability can be guaranteed in spite of parametric uncertainties in the model under the optimization. The so-called fold bifurcation expressing the situation when a real pole crosses the imaginary axis played a crucial role in this research.

D. DDS

Stability issues depended on the value of $\boldsymbol{\tau}$ can be investigated using several methods and techniques. Two basic families of DDS methods for computing the delay stability margins prevail in the literature; namely, time-domain indirect and frequency-domain direct methods. In this survey (dealing inherently with the latter group), research results utilize the following methods for the delay-margin computation:

(i) CTCR;
(ii) Direct method;
(iii) Argument principle (Cauchy theorem) method;
(iv) Schur-Cohn criterion;
(v) Kronecker sum and matrix pencil methods;
(vi) Lyapunov matrix (Kronecker multiplication) approaches;
(vii) Other numerical, semi-analytic and analytic methods.

The key idea lies in the determination of all stability switching system poles (i.e. the characteristic quasipolynomial zeros) located exactly on $\mathbb{C}^0$, which can be used to determine the stability margin. In fact, only the rightmost subset of the spectrum makes the system switch from stability to instability or vice versa. Techniques included in all the above items (except for (iii)) are based on the elimination of exponential terms from $\Delta(s)$; however, only bivariate polynomials include the information of the critical delay values explicitly. This can be done by the direct replacement of the exponential terms by using, e.g., the Möbius or Rekasius transformation (substitution) according to (48) and (49). A common alternative way is to apply the half-angle tangent substitution as follows:

$$\begin{aligned}
&\exp(-\tau j\omega) \rightarrow \cos(\upsilon) - j\sin(\upsilon),\ \upsilon = j\omega,\\
&\cos(\upsilon) = \frac{1-z^2}{1+z^2},\ \sin(\upsilon) = \frac{2z}{1+z^2},\ z = \tan\left(\frac{\upsilon}{2}\right),\\
&\tau_k = \omega^{-1}\left(2\tan^{-1}(z) + k\pi\right),\\
&\qquad k \in \mathbb{Z},\ \tan^{1}(\cdot) \in [0, \pi). \qquad (50)
\end{aligned}$$

1) CTCR

Some research results have extended or improved the original CTCR concept. Sipahi and Delice [165] focused on the so-called core hypersurfaces and showed some their features for the case when $n_\tau > 0$. The core hypersurfaces mean the image of $\wp_0(\boldsymbol{\tau})$ computed from the corresponding multivariate algebraic polynomial $p(\omega, \mathbf{T})$ in the parameter space of pseudo-delays. The authors were concerned with the identification of the asymptotic directions of the delays on the potential stability switching hypersurfaces approaching infinity. These results can also be used in connection with [166] to study strong DIS by covering both finite and infinite delays.

Jesintha Mary and Rangarajan [167] applied the resultant theory introduced in [165] in order to investigate a new flexible methodology for stability analysis of in load frequency control scheme with delays in the transmission of control signals from the control center to generating unit. The proposed method offered larger delay margin and takes less computation time compared to some existing methods.

A recent work of Sipahi's [168] utilized the knowledge (based on the above-introduced research) that it is possible to compute the exact range of the imaginary spectrum of such systems to design imaginary poles with the objective to manipulate stability regions in the delay space. In addition, Kammer and Olgac [169] studied stability of dRTDSs via the CTCR paradigm by means of the equivalence of a general class of distributed delay system to a discrete-delay system with multiple independent delays.

A comparison between delay space (represented by $\wp_0(\boldsymbol{\tau})$, $\wp_{off}(\boldsymbol{\tau})$) and the spectral delay space was presented in [170]. The latter domain contains pointwise frequency information as well as the delay and it was preferred here for its advantageous boundedness properties and the simple construction of stability transition boundaries.

Gao and Olgac [171]–[173] investigated the bounds of the imaginary spectra via the substitution (49) and by deploying the Dixon resultant theory [174] for a RTDS with an arbitrary number of delays. Consequently, the proof of the differentiability of the crossing-frequency variations $d\omega/dz_i$ was provided to investigate the bounds. As a result, 2D cross-sections of the hypersurfaces were extracted. The concept of the so-called 3D building blocks in the spectral delay space [175] was utilized to meet this objective.

The exact delay bound for a consensus of linear multi-agent systems with a fixed and uniform communication time delay was determined by Cepeda-Gomez in [176] in an efficient manner by using the CTCR methodology. A state transformation was performed to decouple the system and simplify the problem prior to the stability analysis.

2) DIRECT METHOD

The "direct" refers to the method introduced by (19)-(22). Sönmez *et al.* [177] studied the DDS problem for load frequency control systems with constant communication delays of the commensurate degree of one and two. However, the complete stability windows were not considered because only the minimum positive value of (22) with $RT = +1$ was taken as the unique delay margin.

3) ARGUMENT PRINCIPLE METHOD

This definite integral stability method, originated from the argument principle (or the Cauchy theorem), is effective because it only requires a rough estimation of the testing integral over a finite interval to judge DDS. Consider the so-called testing integral $F(\Omega, a)$ defined in Theorem 2. Xu and Wang [106] proved that if $\Delta(s)$ of a NTDS has no imaginary roots and the condition (36) is satisfied, then there exists a sufficiently large $\Omega_0 > 0$ so that for all $\Omega > \Omega_0$ it holds that

$$\mathcal{N} \in \left(-\frac{F(\Omega,0)}{\pi} + \frac{n-1}{2}, -\frac{F(\Omega,0)}{\pi} + \frac{n+1}{2}\right) \quad (51)$$

where $\mathcal{N}$ is the number of poles in $\mathbb{C}^+$. Two DDS algorithms, for finding the parameter (delay)-dependent critical upper limit and a parameter (delay)-independent upper limit without any restriction on the number of time delays, were presented by Xu *et al.* [178] who proved the following theorem.

Theorem 4: Assume that $\Delta(s)$ has no roots on $\mathbb{C}^0$ and (36) holds. Let $\Omega_0(\boldsymbol{\tau}) = \max(\omega_R, 0)$ where ω_R stands for the maximal positive root of $\mathbb{R}(\omega) := \mathrm{Re}\left(j^{-n_\Delta}\Delta(j\omega)\right)$. Then (37) and (51) are true for all $\Omega > \Omega_0$.

4) SCHUR-COHN METHOD

Mulero-Martínez [179] presented a modified Schur-Cohn criterion for RTDSs with commensurate delays that requires seeking real roots only, which is comparable to the Rekasius substitution criterion. In contrast to the classical Schur-Cohn criterion, the approach is based on the application of triangular matrices over a polynomial ring in a similar way as in the Jury test of stability for discrete systems, and it halves the dimension of the subjected polynomial. It starts with the construction of a bivariate polynomial $r \in \mathbb{C}[\omega, z]$, $r(\omega, z) = \sum_{i=1}^{n_C} b(\omega) z^i$ from $\Delta(s) = \sum_{i=0}^{n_C} d(s)\exp(-shi)$ where $b(\omega) = d(j\omega)$, $z = \exp(-sh)$, n_C is the commensuracy degree, and h stands for the base delay. Then, two associated triangular matrices are assembled, from which the determinant polynomial $\zeta(\omega)$ is calculated. As a core of the approach, the following theorem holds

Theorem 5: Let $\omega_c > 0$ be a real root of $\zeta\left(\sqrt{s}\right)$, then $\pm \mathrm{j}\sqrt{\omega_c}$ is a pair of poles of the RTDS.

5) KRONECKER SUM AND MATRIX PENCIL METHODS

Ma *et al.* [180] studied DDS of a NTDS with a single delay

$$\mathbf{H}_0\dot{\mathbf{x}}(t) + \mathbf{H}_1\dot{\mathbf{x}}(t-\tau) = \mathbf{A}_0\mathbf{x}(t) + \mathbf{A}_1\mathbf{x}(t-\tau) \quad (52)$$

by applying the matrix pencil and the linear operator methods. The main result of the method regarding the DDS problem was enshrined in the following theorem.

Theorem 6: If s_k is a purely complex root of $\Delta(s)$, it is also a zero of

$$\Delta_{ass,2}(s) := \det\begin{pmatrix} (s\mathbf{H}_0 - \mathbf{A}_0) \otimes (s\mathbf{H}_0 + \mathbf{A}_0) \\ -(s\mathbf{H}_1 - \mathbf{A}_1) \otimes (s\mathbf{H}_1 + \mathbf{A}_1) \end{pmatrix}, \quad (53)$$

see also Louisell [77].

6) LYAPUNOV MATRIX APPROACHES

Consider the system (23) again, one approach is based on the fact that any purely imaginary root of $\Delta(s)$ is also a root of the polynomial

$$\Delta_{ass,3}(s) := \det\left(\left(s\mathbf{I} + \mathbf{A}_0^T\right) \otimes (s\mathbf{I} - \mathbf{A}_0) - \left(\mathbf{A}_1^T \otimes \mathbf{A}_1\right)\right) \quad (54)$$

that is also the characteristic polynomial of the system

$$\begin{aligned} \mathbf{X}'_0(\theta) &= \mathbf{X}_0(\theta)\mathbf{A}_0 + \mathbf{X}_{-1}(\theta)\mathbf{A}_1 \\ \mathbf{X}'_1(\theta) &= -\mathbf{A}_1^T\mathbf{X}_0(\theta) - \mathbf{A}_0^T\mathbf{X}_{-1}(\theta) \end{aligned} \quad (55)$$

see (53) for the comparison. System (55) can be then subjected to the computation of Lyapunov matrices. Once the spectrum of (55) is computed, critical values of the delay can be obtained by substituting these roots into $\Delta(s)$. Ochoa *et al.* [181] adopted the above idea to derive explicit relations between the spectrum of an original dRTDS and NTDS, and that the delay-free system (55), which constituted a bridge between time-domain and spectral approaches. Delay-dependent stability regions were determined as well. Since $\Delta_{ass,3}(s)$ has only even powers of s, the searching of imaginary poles was reduced to the computation of real roots of $\Delta_{ass,3}(\lambda)\big|_{\lambda=s^2}$. To solve this task, the authors utilized Sturm's theorem that is based on the computation of sign changes of the Sturm sequence.

Another technique based on the Lyapunov–Krasovskii methodology to investigate delay-dependent (robust) exponential stability of a RTDS was derived by Cao [182]. The author used LMIs and slack matrices to get the upper bound of the exponential decay rate. The given criterion provides the computation method of the value of $L_{\max}$, so that the system is (robustly) exponential stable for $L \in (0, L_{\max}]$, i.e. no other stability windows were considered. A comparison with some other methods was also given to the reader. Sun *et al.* [183] derived the sufficient condition for the delay-dependent asymptotic stability of the closed-loop power system with prescribed degree of stability α (i.e., the decay rate or a spectral abscissa) based on the Lyapunov stability theory and transformation operation in complex plane, and presented a method based on LMIs to calculate the delay margin of the closed-loop system considering the prescribed value of α.

7) OTHER METHODS

Regarding the research results already introduced in this survey, Wang *et al.* [146] calculated mutual delay values satisfying exponential stability of a RTDS with example case studies supporting their method. Complete stability intervals for the base delay of a system with commensurate delays were determined by means of the frequency sweeping method and the Puiseux series in the works of Li *et al.* [151]–[154].

The singular value decomposition technique was used by Ramachandran and Ram [184] to determine critical delays of a single-input multi-output (SIMO) system. The leading idea was as follows: Let $\Delta(s) = \det(\mathbf{A} - s\mathbf{B} + \exp(-s\tau)\mathbf{H})$ where $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{2n\times 2n}$, and $\mathbf{H}$ is a rank-one matrix subject to the singular value decomposition $\mathbf{H} = \mathbf{U\Sigma V}$, $\mathbf{\Sigma} = diag\left(\sigma \ 0 \dots 0\right)$. After some algebraic operations, the condition $\Delta(s)|_{s=\mathrm{j}\omega} = 0$ can be expressed as

$$N(s_k)\bar{N}(s_k) - D(s_k)\bar{D}(s_k) = 0 \quad (56)$$

where $N(s_k) = \det\mathbf{Q}$, $D(s_k) = \sigma\det\mathbf{Q}_1$, $\mathbf{Q} = \mathbf{U}^T(\mathbf{A} - s_k\mathbf{B})\mathbf{V}$, $\mathbf{Q}_1$ stands for the $(2n-1)\times(2n-1)$ trailing submatrix of $\mathbf{Q}_1$, s_k is a purely imaginary root, and the bar expresses the complex conjugate, i.e. the problem is reduced to the task of finding the roots of a polynomial. Nevertheless, the technique cannot be used for a multi-input multi-output (MIMO) system. In this case, the authors separated the matrix eigenvalue problem into its real and imaginary components, so that the problem of determining the critical delay was transformed to

$$\mathbf{P}(\tau, s_k)\mathbf{z} = 0 \quad (57)$$

where s_k is a purely imaginary repeated eigenvalue with a multiplicity larger than one. Since the Jacobian matrix associated with (57) is singular in the neighborhood of the solution, the convergence of Newton's method is linear; hence, a bisection algorithm for solving the problem was developed.

Pontes Duff *et al.* [185] solved the model reduction problem (45) for RTDSs with multiple delays via the so-called TF-IRKA algorithm [186] giving rise to the finite-dimensional model

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t). \quad (58)$$

The obtained model was then used to estimate stability regions.

An interesting comparative study on the stability analysis of DCPPS was presented by Gao *et al.* [187] where three methods were adopted; namely, a Padé approximation based method [123], the explicit infinitesimal generator discretization-based method [84] and a DDS technique [188] that allows for the determination of the maximal delay such that a DCPPS remain stable by using an LMI technique (see [182] for the comparison). The conservativeness of the DDS method was analyzed via an example and it was inter alia observed that the Padé approximation based method is sufficiently accurate even for a low approximation order.

A simply implementable gridding delay-discretization DDS technique was published in [189] and [190]. This numerical concept is based on the recursive approximation of $\Delta(s)$ by the associate polynomial in the vicinity of the current estimation of the rightmost pole s_0 in every node of the grid in the delay space. While in [189], the associated polynomial $\Delta_{ass,4}(s|s_0)$ was obtained from the Taylor series expansion, the bilinear transformation together with the pre-warping technique were utilized in [190] to get $\Delta_{ass,5}(z|s_0, T_s)$ where T_s expresses the sampling time. The leading zero of $\Delta_{ass,4}(s|s_0)$ or $\Delta_{ass,5}(z|s_0, T_s)$ then yielded the eventual rightmost pole estimation for the next grid node. The estimation of the switching poles was further enhanced by using the average of *RT* values (a combined Newton's technique) and by the linear interpolation, respectively. The concept clearly utilizes the root continuity property (see item (v) of Proposition 1); however, one has to be careful in the neutral delay case (Proposition 2 (iv), Proposition 3).

A combination of three techniques to determine the delay stability margin for wide-area measurement systems (WAMS) – modeled by RTDSs with multiple delays – was proposed in [191]. Namely, matrices $\mathbf{A}_i$ in (1) are standardized into the Jordan form first, yielding a new state vector $\mathbf{z}(t)$. Second, the Taylor expansion is applied to separate the connection between $\mathbf{z}(t)$ and $\mathbf{z}(t-\tau_i)$. Finally, the Schur simplification [192] is implemented to reduce the number of state variables.

Roales and Ródriguez [193] studied the existence of stability switches and Hopf bifurcations (i.e. the periodic stability boundary) for the second-order scalar delay differential equation $\ddot{x}(t) + a\dot{x}(t-\tau) + bx(t) = 0$, $t, \tau > 0$, in which $a, b \in \mathbb{C}$. The presented analytic derivations were based on the theorem established in [194] that characterizes, for the critical values τ_i such that $\Delta(\mathrm{j}\omega, \tau_i) = 0$, the variation of the number of zeros with nonnegative real parts of $\Delta(s, \tau)$ in terms of the order and sign of the first nonzero derivate of $F(\omega) := |\mathrm{Re}(\Delta(\mathrm{j}\omega))|^2 - |\mathrm{Im}(\Delta(\mathrm{j}\omega))|^2$.

E. DIS

Frequency-based DIS methods are generally built on the verification of the non-existence of purely imaginary system poles for arbitrary delay values. This task is usually achieved by transforming $\Delta(s)$ into associated (auxiliary) polynomial $\Delta_{ass}(s)$ or $\Delta_{ass}(z)$, which is completely free of delays and can be uni-, bi- or even multivariate, and then by proving that there is no zero of $\Delta_{ass}(s)$ lying exactly on the imaginary axis, or no zero z_k of $\Delta_{ass}(z)$ such that $z_k \in \mathbb{T}$.

Delice and Sipahi [166] used the technique of computing the resultant and consequently that of the iterated discriminant [68] to eliminate pseudo-delays from $p(\omega, \mathbf{T})$ (see the description of the CTCR concept above), which allowed one to construct a single-variable function $D(\omega)$ to be equal to zero. Then, the non-existence of any positive real root of $D(\omega)$ - which is a sufficient DIS condition - was proved by the Déscartes rules of signs. However, infinite delays were omitted in this technique. Asymptotic directions of the delays on the potential stability switching hypersurfaces approaching infinity derived by Sipahi and Delice [165] linked DDS and the strong DIS problem (here, the authors used the term "strong" for DIS including infinity delays).

Concerning multiple-delay cases, the comprehensive study by Nia and Sipahi [195] also utilized the Rekasius transformation (29) and the resultant theory to investigate DIS in the delay space and the controller parameters space for active vibration control systems. In addition, Sturm sequences were applied to establish the necessary and sufficient conditions in identifying the number of distinct positive real roots of $D(\omega)$.

A matrix pencil methodology along with an algebraic method were utilized by Ma *et al.* [180] to investigate the DIS problem via $\Delta_{ass,2}(s)$ as in (53).

Ergenc [196] presented a method for determining the DIS zones of a general RTDS with multiple delays against parametric uncertainties. This method adopted the Kronecker summation scheme $\Delta_{ass,1}(z)$ as in (24) expressed by means of the Kronecker multiplication operators. The system is DIS if

$$\mathrm{Re}(s : \Delta(s, \mathbf{p}) = 0) < 0 \tag{59}$$

and all zeros z_k of $\Delta_{ass,1}(z, \mathbf{p})$ satisfy $z_k \notin \mathbb{T}$ where $\mathbf{p}$ represents a vector of unknown parameters. In fact,

$$\Delta_{ass,1}\left(z_1, z_2, \ldots, z_{n_\tau}, \mathbf{p}\right) = \sum_{j=1}^{m} b_j\left(z_1, z_2, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{n_\tau}, \mathbf{p}\right) z_i^j$$

is a self-inversive (symmetric) multivariable polynomial satisfying $\Delta_{ass,1}(z_i) = z_i^{n_\tau}\overline{\Delta_{ass,1}(1/\overline{z_i})}$ for $i = 1, 2, \ldots, n_\tau$. The following unique property of self-inversive polynomials was utilized: $\beta = (2\mu + 1) - m$ where β is the number of its zeros lying on $\mathbb{T}$, and μ is the number of zeros inside $\mathbb{D}$ (including multiplicity). With the combination of this property and another general polynomial property (Pellet's theorem), the following sufficient condition for DIS was presented:

Theorem 7: The system is DIS if (59) holds and

$$\left|b_\mu\left(z_1, z_2, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{n_\tau}, \mathbf{p}\right)\right| > \sum_{j=1, j\neq\mu}^{m} \left|b_j\left(z_1, z_2, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{n_\tau}, \mathbf{p}\right)\right| \tag{60}$$

for $\mu \leq m/2 - 1$, $i = 1, 2, \ldots, n_\tau$.

An experimental study verifying this result was presented in [197]. The methodology was further improved by Alikoç and Ergenc [198] where the Bistritz tabulation method [199] was used to determine the location of zeros with respect to the unit circle for a single delay RTDS. The method is based on a three-term recursion of symmetric polynomials and the number of sign variations of these polynomials at $z = 1$; namely, the sign variation in a sequence of numbers obtained by the solution of recursive equations calculated from the polynomial $D(z) = \mathrm{d}\Delta_{ass,1}(z)/\mathrm{d}z$ is evaluated instead of the condition (60), which enables the use of real arithmetic operations. Alikoç and Ergenc [200] extended this technique to multiple incommensurate delays. These results can be utilized, for instance, when determining the controller parameters' set robustness with respect to delay values.

Recall that the DIS problem was investigated also by the semi-discrete approximation of $\mathcal{A}$ by means of linear spline functions presented by Fabiano [89].

The argument principle (or, contour integral) method [107], [178], was used to deal with the DIS as well. Consider Theorem 4, in which ω_R is the maximal positive root of $R_L(\omega)$ instead of $R(\omega)$. Polynomial $R_L(\omega)$ is constructed from $R(\omega)$, the coefficients of which are substituted by their infima that are independent of delay values.

Assuming the reign of spectral methods, marginal yet interesting results were presented by Li *et al.* [201] where the strong DIS condition via LMIs was analyzed using frequency domain discretization into several sub-intervals and the piecewise constant Lyapunov matrices. A series of proposed stability criteria yield necessary and sufficient strong DIS conditions for RTDSs with a single delay which is less conservative than some typical sufficient LMI conditions. It is worth noting that the notion of strong DIS introduced there is rather different than that in [165]. Namely, consider a RTDS with commensurate delays and the base delay h, the system is strongly DIS if $\Delta(s, z) \neq 0$ for all $s \in \mathbb{C}^+$ and $z \in \mathbb{D}$ where $z = \exp(-sh)$. This property is robust against perturbations of parameters in the state matrices in (1), see [202] for details.

F. PARAMETER-DEPENDENT STABILITY

By parameter-dependent stability we mean the stability investigation with respect to system parameters except for delays, i.e., in the non-delay parameter space.

Recalling research results already introduced above again, Dong *et al.* [103] evaluated the optimal parameters for the controller design by searching the global minimum of the spectral radius of the transition matrix that was obtained by means of the DQ method. In order to solve such optimization problems using gradient descent algorithms, the gradient of the spectral radius of transition matrix with respect to the concerned parameters was analytically formulated.

The Vandermonde/Birkhoff matrix DDS approach for multiple purely imaginary poles by Boussaada and Niculescu has also included non-delay parameters while studying parameter-dependent exponential stability [148], [149].

Otten and Mönnigmann [164] proposed an optimization scheme that was based on the solution of the H_2 minimization problem in the parameter space subject to the manifold of the critical parameter values. In addition, the normal vector has to be solved to enforce a robust distance between any candidate optimal steady state and the critical manifold.

Argument principle based DDS methodology by Xu *et al.* [178] can be applied to stability analysis with respect to non-delay parameter values as well.

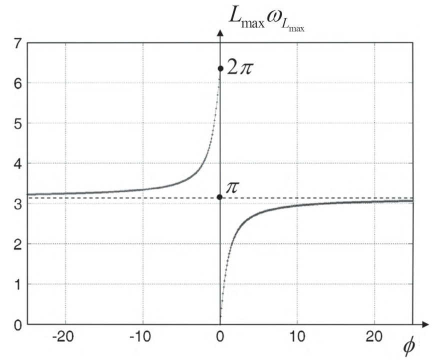

FIGURE 3. Function $\phi \mapsto L_{\max}\omega_{L_{\max}}$ [204].

A simple systematic frequency sweeping procedure for solving the exact stability boundaries in the parameter plane $\mathbf{p} = (p_1, p_2)$ for RTDSs was proposed by Perng [203]. Note that the methodology has also been used to solve the DDS problem for systems with commensurate delays as follows:

Let $\exp(-sh) = \exp(-\mathrm{j}\omega h) = \cos(\omega h) - \mathrm{j}\sin(\omega h) = p_1 - \mathrm{j}p_2$, then after some algebraic operations on goniometric functions, the potential stability boundary plots in $p_1 - p_2$ space can be obtained again. Since it must hold that $|\exp(-sh)| = 1$, the boundary must intersect the unit circle in the parameter plane for admissible solutions. The exact maximum delay value for asymptotic stability then reads $h = \omega^{-1}\cos^{-1}(p_1) = \omega^{-1}\sin^{-1}(p_2)$. If no intersection is found, the system is DIS (or unstable).

The design of parameters $\mathbf{p} = (p_1, p_2)$ such that the system represented by (23) is asymptotically stable for $L \in (0, L_{\max}]$ with a predetermined (known and fixed) value $L_{\max}$ was presented by Sipahi [204]. The author used the Rekasius substitution and introduced the sweeping parameter $\phi = \omega\mathbb{T}$ in an interval $\phi \in [\phi_{\min}, \phi_{\max}]$. Then, it can be computed from (48) that $\exp\left(-\mathrm{j}\omega_{L_{\max}}L_{\max}\right) = (1 - \mathrm{j}\phi)/(1 + \mathrm{j}\phi)$ to get a polynomial $p\left(\omega_{L_{\max}}, \phi, \mathbf{p}\right)$ instead of the quasipolynomial $\Delta\left(\mathrm{j}\omega_{L_{\max}}, \mathbf{p}\right)$ where the corresponding frequency reads

$$\omega_{L_{\max}} = \frac{2}{L_{\max}}\left(\tan^{-1}(\phi) - (\operatorname{sgn}(\phi) - 1)\frac{\pi}{2}\right),$$
$$\phi \neq 0, \quad 0 < \omega_{L_{\max}} \leq 2\pi/L_{\max}, \tag{61}$$

see Fig. 3. Then, for these fixed values, one should simultaneously solve the set of equations $\operatorname{Re}(p(\mathbf{p})) = \operatorname{Im}(p(\mathbf{p})) = 0$. However, the feasibility of this solution must be verified by the computation of the number of imaginary crossing, M, by means of Theorem 1.

Hence, it is necessary to compute the eigenvalues of Ω_Π as in Theorem 1, followed by the verification where these values are included in the system spectrum Ω. Note that $M \leq n^2$ (for $\Omega > 0$) and in the referred research study, the author enforced $M = 1$ initially.

Schrödel *et al.* [205] presented a comparative overview of four existing frequency-based methods for the stability boundary calculation problem in the parameter space, namely, the Rekasius substitution method, the direct method [72], the Kronecker multiplication method [77] (see also Ma *et al.* [180]) and the so-called matrix sum method [74]. The last one of the methods is based on the elimination of ω from the characteristic equation and the solution of the associated equation for $z \in \mathbb{T}$. The characteristic equation can be rewritten as

$$\Delta_{ass,4}(s, z) := \det((s\mathbf{I} - \mathbf{M}(z))) = 0 \tag{62}$$

where $z = \exp(-sh)$ and $\mathbf{M}(z)$ is apparent from (2) for commensurate delays. Crossing poles satisfy $z_c \in \mathbb{T}$ and $s_c = \pm \mathrm{j}\omega_c$.

Equation (62) can also be reformulated as $\det((z\mathbf{U} - \mathbf{V})) = 0$ where $\mathbf{U}$, $\mathbf{V}$ are matrices that include Kronecker sum and multiplication operations (see [205] for more detail). By solving this equation for $z \in \mathbb{T}$, the crossing frequencies can be obtained from $\Delta_{ass,4}(\omega) = \Delta_{ass,4}(\mathrm{j}\omega, z_c) = 0$ and the corresponding delays via $\tau_k = \omega^{-1}(\arg(z) + 2k\pi)$, $k \in \mathbb{Z}$.

In [205], a generalization of the problem of calculating the stability region for TDSs in the delay and non-delay parameter space (which is very close to the CTCR paradigm) was also given to the reader. In addition, three types of stability boundaries were introduced.

It is worth noting that most of the results on parameter-dependent stability were obtained for control systems and related tasks of controller parameters tuning, see e.g. [119], [124], [125], [146], [165], [195], which goes, however, beyond the objective of this survey that is aimed at system analysis.

Selected results from the general part of this section (and also from the previous one in some cases) are summarized in Table 2 to provide the reader with an overview of the theoretical stability studies.

VII. ENGINEERING APPLICATIONS AND CASE STUDIES

This section is focused on the commented list of academic or practical applications of the methods for LTI-TDSs spectrum analysis. Note that this section is summarized in Table 3.

Method approximating $\mathcal{A}$ or $\mathcal{T}(t)$, introduced herein in sections from IV to VI, can be found in the literature as favorite tools for the analysis of milling processes – unfortunately, these models of type (23) usually include a time-dependent state matrix $\mathbf{A}_1$.

Recall that Tang *et al.* [98] predicted milling stability via an improved FD method with Lagrange polynomial interpolation, and the authors presented a comparison with SD and NI techniques as well. The same problem was solved using FD and NI methods in [95] and [99], respectively.

The DQ method utilized for the stability analysis of milling processes was presented by Ding *et al.* [102].

Quo *et al.* [207] utilized the third order FD method to get the exact stability bounds. Ozoegwu [208] presented a method being very close to the FD one, yet the least squares (also called the hyper third-order) approximation was applied instead of the interpolation procedure.

The hyper third-order approximation followed by the NI method was further used by Ozoegwu *et. al.* [209] and extended third and fourth order vector NI schemes for one-degree-of-freedom and two-degree-of-freedom milling processes in [100]. The same authors also presented the use of the SE method while analyzing the chatter stability of a three-tooth plastic end-milling CNC machine [210].

These approximation methods, however, have been applied in other industrial applications as well. Khasawneh [211] utilized the SE method with the barycentric Lagrange formula to analyze the stability of machining processes which may lose stability due to chatter vibrations, i.e., self-excited vibrations due to the surface regeneration effect. A short (in its form) yet comprehensive (in its content) overview of numerical techniques that are based on a finite dimensional approximation of the infinite dimensional system used for the stability prediction of machining processes was presented by Insperger *et al.* [11]. This type of chatter occurs due to workpiece rotations or dynamic cutting load changings.

Kishor *et al.* [8] discussed stability analysis using spectral discretization of time-delayed electric power systems, namely, the 4-generator and the 14-generator Southeast Australian power systems.

The PsC method was used to get the discrete mapping there, and the authors computed the rightmost poles and the spectral abscissa over a wide range of time delays, which characterizes a partial solution of the DDS problem. As introduced above, Ye *et al.* proposed an iterative PsC method for spectral analysis of large DCPPS to overcome computational problems with sparse matrix approximation of the infinitesimal generator [84], [86], and the solution operator [87].

Milano [138], and Milano and Anghel [144] used the PsC technique to compute poles of a large DCPPS and compared it with some other discretization schemes to get a finite-dimensional approximation of the solution operator $\mathcal{T}(t)$. For these results, see also Table 1.

The pseudospectral method by Breda *et al.* [58] was utilized by Coelho *et al.* [212] to analyze the spectrum of a single delay RTDS expressing the feedback control system for an islanded microgrid composed of two or more voltage source inverters with communication delays.

Sensitivity analysis of the poles was conducted by Zhao [213] in order to reveal the dynamic stability margin and to identify the proper range of the control parameters, for an islanded medium-voltage microgrid placed in the Dongao Island. Unfortunately, the authors did not refer to the used method.

Dong *et al.* [214] proposed a stability analysis method of the hybrid energy storage systems with delays and applied it to a lab-scale DC microgrid. The stability margin (i.e., the maximum stabilizable delay) was computed by the determination of purely imaginary poles. The leading idea of the critical poles computation is based on the assumption that all delays are rational numbers or they can be approximated by the rational numbers. Then, one can rewrite the characteristic equation $\Delta(\mathrm{j}\omega_c) = 0$ so that its solution has a period of 2π.

TABLE 2. Stability studies related to pole loci – theory (Section vi).

Authors	Model	Methodology/technique	Problem type
Bonnet *et al.* [51]	NTDS, commensurate delays	Semi-analytic method	H_∞ stability
Fabiano [90]	Scalar NTDS, multiple delays	Approximation of A via spline functions	DIS
Tweten *et al.* [96]	Autonomous and time-periodic RTDS	SD, SE, SLT (comparison)	Finite-dimensional approximation, asymptotic stability
Lehotzky and Insperger [97]	Digitally controlled RTDS	SD	Finite-dimensional approximation, exponential stability
Zhang *et al.* [101]	(d)RTDS (Mathieu equation with multiple delays)	NI, SD	Finite-dimensional approximation, exponential stability
Dong *et al.* [103]	NTDS	DQ	Finite-dimensional approximation, exponential stability, strong stability, parameter-dependent stability
Chen and Liu [104]	RTDS (Linearized system at the equilibrium)	Contour integral method	Local asymptotic stability of the positive equilibrium for the Lotka-Volterra system
Xu and Wang [106]	NTDS	Mikhailov criterion (argument principle, Cauchy theorem), Nyquist plot	Exponential stability, asymptotic stability
Xu *et al.* [107]	NTDS, RTDS, multiple delays	Mikhailov criterion (Cauchy theorem)	Exponential stability, asymptotic stability, DDS, DIS, computing the spectral abscissa
Xu *et al.* [178]	NTDS, multiple delays	Mikhailov criterion (Cauchy theorem)	Exponential stability, asymptotic stability, DDS, DIS, parameter-dependent stability
Nguyen *et al.* [117], [118]	NTDS, commensurate delays	Semi-analytic method	H_∞ stability for multiple chains of poles asymptotic to the same point on $\mathbb{C}^0$
Rabah *et al.* [119]	Mixed (d)RTDS and NTDS, possibly singular	Riesz basis, resolvent boundedness approach	Asymptotic non-exponential and strong stability
Avanessoff *et al.* [120]	RTDS, NTDS, commensurate delays	Semi-analytic method, Padé-2 approximation	H_∞ stability
Milano [138]	RTDS (DCPPS), multiple delays	PsC, Runge-Kutta and linear multi-step discretization	Small signal asymptotic stability, computing the rightmost spectrum
Damak *et al.* [141]	Linear difference equations with commensurate delays	Lyapunov–Krasovskii and spectral methods	Asymptotic stability
Zhang and Sun [142]	RTDS	PsC, SD, Lyapunov theory	Asymptotic stability
Milano and Anghel [144]	RTDS (DCPPS), single delay	PsC	Small signal asymptotic stability, computing the rightmost spectrum
Domoshnitsky *et al.* [145]	Scalar second order RTDS, multiple delays	W-transform	Exponential stability, DDS
Wang *et al.* [146]	RTDS, NTDS	Extended Hermite-Biehler theorem	Computing the rightmost poles on $\mathbb{C}^0$
Boussaada and Niculescu[147], [149]	RTDS, two delays	Vandermonde and Birkhoff matrices	Exponential stability, bifurcation analysis, multiplicity of poles on $\mathbb{C}^0$, DDS, parameter-dependent stability
Louisell [150]	RTDS	Matrix operator approach	Asymptotic stability, computing purely imaginary poles
Li *et al.* [151]-[154]	RTDS, commensurate delays	Frequency sweeping, Puiseux series expansion	Asymptotic stability, simple and multiple poles on $\mathbb{C}^0$, DDS
Cai *et al.* [155]	RTDS, commensurate delays	Weierstrass preparation theorem, Puiseux-Newton diagram	Asymptotic stability, simple and multiple poles on $\mathbb{C}^0$
Méndez-Barrios *et al.* [156]	RTDS, commensurate delays	Weierstrass preparation theorem, Weierstrass preparation theorem, Puiseux series	Asymptotic stability, multiple roots on $\mathbb{C}^0$
Bouzidi *et al.* [157]	RTDS, commensurate delays	Möbius transformation, rational univariate representation, Puiseux series expansion	Asymptotic stability, simple and multiple roots on $\mathbb{C}^0$
Gu *et al.* [159], Irofti *et al.* [160]	RTDS, two delays	Analytic calculations	Asymptotic stability, double roots on $\mathbb{C}^0$
Irofti *et al.* [161]	RTDS, two delays	Analytic calculations	Asymptotic stability, triple and quadruple roots on $\mathbb{C}^0$

TABLE 2. ***(Continued.)*** **Stability studies related to pole loci – theory (Section vi).**

Authors	Model	Methodology/technique	Problem type
Abusaksaka and Partington [162]	NTDS, single delay	Semi-analytic method	H_∞ and BIBO stability, "small" modulus poles
Otten and Mönnigmann [164]	RTDS, multiple delays and uncertain parameters	Analytic H_2 optimization	Asymptotic stability, fold bifurcation
Sipahi and Delice [166]	RTDS, multiple delays	CTCR, analytic calculations	DDS, DIS
Jesintha Mary and Ragarajan [167]	RTDS, multiple delays	CTCR, Rekasius substitution, resultant theory	DDS in load frequency control system
Kammer and Olgac [169]	dRTDS	CTCR	DDS
Gao *et al.* [170]	RTDS	CTCR, spectral delay space	DDS, comparison of delay space vs. spectral delay space
Gao and Olgac [171]-[173]	RTDS, multiple delays	CTCR, 3D building blocks, Dixon resultant theory	DDS, 2D cross-sections of the hypersurfaces
Cepeda-Gomez [176]	RTDS (communication delays)	CTCR	DDS
Sönmez *et al.* [177]	RTDS, commensurate delays	Direct method	DDS in load frequency control systems
Mulero-Martínez [179]	RTDS, commensurate delays	Schur-Cohn criterion, bivariate polynomial, triangular matrices	DDS
Ma *et al.* [180]	Singular NTDS, single delay	Matrix pencil method, linear operator method	DDS, DIS
Ochoa *et al.* [181]	dRTDS, NTDS, multiple delays	Lyapunov matrix approach	DDS, DIS
Cao [182]	RTDS, single delay	Lyapunov–Krasovskii methodology, LMIs	(Robust) DDS
Sun *et al.* [183]	RTDS (interconnected power systems)	Lyapunov stability theory, LMIs, Schur balanced truncation model reduction	DDS (with guaranteed exponential decay rate)
Ramachandran and Ram [184]	RTDS, single delay, commensurate delays	Singular value decomposition, Newton's method	DDS
Pontes Duff *et al.* [185]	RTDS, multiple delays	H_2 approximation, TF-IRKA	DDS
Pekař and Prokop [189]	RTDS, NTDS, multiple delays	Taylor series expansion, linear interpolation	DDS
Pekař *et al.* [190]	RTDS, multiple delays	Bilinear transformation, pre-warping, *RT*	DDS
Dong *et al.* [191]	RTDS (WAMS), multiple delays	Jordan form, Taylor series expansion, Schur simplification	DDS
Roales and Ródrigez [193]	Second order RTDS, single delay	Analytic calculations	DDS
Delice and Sipahi [166]	RTDS, multiple delays	Déscartes rule of signs, iterated discriminant and resultant theory	DIS
Nia and Sipahi [195]	RTDS, multiple delays	Rekasius substitution, resultant theory, Sturm sequences	DIS
Ergenc[196], Ergenc and Alikoç [197]	RTDS, multiple delays	Kronecker multiplication, self-inversive polynomial, Pellet theorem,	DIS
Alikoç and Ergenc [198]	RTDS, single delay	Kronecker multiplication, self-inversive polynomial, Bistritz tabulation algorithm	DIS
Alikoç and Ergenc [200]	RTDS, multiple delays	Kronecker multiplication, self-inversive polynomial, Bistritz tabulation algorithm	DIS
Perng [203]	RTDS, commensurate delays	Parameter plane method	Parameter-dependent stability, DDS, DIS
Sipahi [204]	RTDS, single delay, commensurate delays	Rekasius substitution, Kronecker multiplication	Parameter-dependent stability
Schrödel *et al.* [205]	RTDS, commensurate delays	Rekasius substitution, direct method, Kronecker multiplication method, matrix sum method	Parameter-dependent stability

TABLE 3. **Stability studies related to pole loci – applications (Section vii).**

Authors	Model	Methodology/technique	Problem type
Tang *et al.* [98]	RTDS (possibly time-variant)	FD, SD, NI (comparison)	Stability prediction of milling processes
Ding *et al.* [95]	RTDS (possibly time-variant), single delay	FD	Stability prediction of milling processes
Ding *et al.* [99]	RTDS (possibly time-variant), single delay	NI	Stability prediction of milling processes
Ding *et al.* [102]	RTDS (possibly time-variant), single delay	DQ	Stability analysis of milling processes
Quo *et al.* [207]	RTDS (possibly time-variant), single delay	FD	Stability analysis of milling processes
Ozoegwu [208]	RTDS (possibly time-variant), single delay	Hyper third-order least squares approximation (FD)	Stability analysis of milling processes
Ozoegwu [100], Ozoegwu *et al.* [209]	RTDS (possibly time-variant), single delay	Hyper third-order and fourth-order approximation (NI)	Stability analysis of milling processes
Ozoegwu *et al.* [210]	RTDS (possibly time-variant), single delay	SE	Chatter stability analysis of milling CNC machine
Khasawneh [211]	RTDS (possibly time-variant), single delay	SE	Stability analysis of machining processes
Insperger *et al.* [10]	RTDS (possibly time-variant)	Finite dimensional approximation techniques (listing)	Stability prediction of machining processes
Coelho *et al.* [212]	RTDS, single delay	PsC	Computing the spectrum of a microgrid control system with delays
Zhao *et al.* [213]	RTDS, single delay	-	Parameter-dependent sensitivity analysis of pole of an islanded medium-voltage microgrid
Dong *et al.* [214]	RTDS, multiple delays	Delay space re-scaling	Computing the delay margin (or, rightmost poles on $\mathbb{C}^0$) for hybrid energy storage systems in DC microgrids
Petit *et al.* [215]	RTDS, single delay, commensurate delays	Lambert W function	Analytic calculation of the possible onset of Turing instabilities
Yi *et al.* [216]	RTDS, single delay	Lambert W function	Computing the rightmost poles of a delayed neural network
Gölgeli and Özbay [218]	RTDS, two delays	YALTA	Finding allowable delays for local stability (DDS)
Qiao and Sipahi [219]	RTDS with DDC, single delay	TRACE-DDE	DDS with DDC, stability of multi-agent consensus system with communication delays
Sipahi *et al.* [223]	dRTDS, two delays	Taylor expansion, asymptotic behavior analysis	Driver's decision making, DDS, DIS
Ding *et al.* [224]	RTDS, single delay	Multiple time scales method	Stability analysis of a glue dosing process for particleboard, double Hopf and pitchfork bifurcations
Takács and Stépán [225]	RTDS, commensurate delays	D-subdivision	Parameter-dependent stability of lateral vibration of four-wheel vehicles, Hopf bifurcation
Boussaada *et al.* [226]	RTDS, single delay	Vandermonde/Birkhoff matrix approach	Multiplicity of poles on $\mathbb{C}^0$, DDS, parameter-dependent stability for a delayed mechanical system
Alikoç *et al.* [227]	RTDS, two delays	Kronecker multiplication, CTCR	DDS of a train following problem with multiple communication delays
Eris and Ergenc [229]	RTDS, two delays	Kronecker multiplication, CTCR	DDS of a delayed resonator, resonance and stability maps
Olgac *et al.* [231]	RTDS, single delay	CTCR	Combined DDS and parameter-dependent stability of blade/casing rub problem in turbomachinery
Ai *et al.* [232]	RTDS, single delay, two delays	CTCR	DDS of a robotic actuator system, dominant poles loci
Zalluhoglu *et al.* [233]	NTDS, multiple delays	CTCR	Parameter-dependent thermoacoustic stability of a Rijke tube
Gündüz *et al.* [234]	RTDS, single delay	Direct method	Delay margin computation of a micro-grid system with a constant communication delay

TABLE 3. ***(Continued.)*** **Stability studies related to pole loci – applications (Section vii).**

Authors	Model	Methodology/technique	Problem type
Sönmez and Ayasun [236]	RTDS, single delay	Direct method	Delay margin computation of single-area load frequency control system with constant communication time delay
Breda *et al.* [92]	(d)RTDS	Pseudospectral approach	Local asymptotic stability of populations of the Daphnia type
Beretta and Breda [237]	(d)RTDS	Pseudospectral approach	Stability switches of the positive equilibrium for population-growth models
Diekmann *et al.* [238]	RTDS, single delay	Analytic calculation	Computing the complete spectrum and parameter-dependent stability of a cell population model
Ünal *et al.* [240], [241]	RTDS, two delays	DDE-BIFTOOL	Sustained oscillations and stability properties of an Oregonator model (Belousov-Zhabotinskii reaction)

Hence, the interval $[0, 2\pi)$ is discretized, and system poles are then computed in every single discrete step inside this interval.

Although the Lambert W function has a limited utilization due to model restrictions, some engineering applications can be found. For instance, Petit *et al.* [215] studied reaction-diffusion systems with a time delay considered in the complex networks in the framework of Turing instabilities. Explicit analytic conditions for the onset of patterns as a function of the main involved parameters, the time delay, and the network topology were obtained using the scalar Lambert W function. The authors then predicted whether or not the systems would exhibit a wave pattern associated with a Hopf bifurcation, or a stationary Turing pattern. Yi *et al.* [216] used the function to obtain the rightmost poles of neural networks with time delays and parametric uncertainties modeled by a single delay RTDS. However, note also that more particular applications of the Lambert W function have been made concerning controller design, see e.g. [109], [217], and references therein.

Niu *et al.* [123] used the Padé approximation to estimate the spectrum of a power system with a time delay, see Table 1. Gölgeli and Özbay [218] utilized the YALTA software to investigate the unique local stability by analyzing the impact of the nicotine exposure on the cholesterol biosynthesis. The so-called delay-dependent coupling (DDC) was considered in [219] to prevent instability in a multi-agent system in which agents communicate with each other under homogeneous delays, while attempting to reach consensus. The system model has a simplified form as follows

$$\dot{\mathbf{x}}(t) = f(L)\,\mathbf{A}\mathbf{x}(t-L) \tag{63}$$

where $f(L)$ represented the DDC as a function of the delay value L. The main idea while designing the stability of (63) was based on the following formula for the computation of the delay margin $L_{\max}$.

$$L_{\max} = \frac{1}{f(L)} \min_k \frac{\eta_k/2}{2\alpha_k \sin \eta_k/2} \tag{64}$$

where α_k, η_k are related to the particular eigenvalue s_k of $\mathbf{A}$ according to Fig. 4.

Trajectories of poles were obtained via TRACE-DDE tool [220]. Note that a multi-agent consensus dynamics under

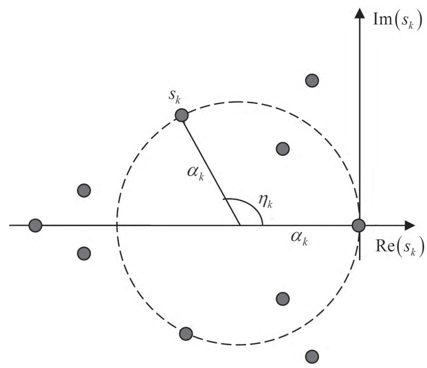

FIGURE 4. **Eigenvalues s_k of A with respect to α_k, η_k [219].**

a communication delay, the delay margin and the network topologies to reduce the duration to reach consensus were also investigated by means of the rightmost poles e.g. in works [221], [222], which, however, can be considered as control rather than analytic tasks.

Sipahi *et al.* [223] studied how the memory of drivers modeled by distributed delays affects the decision-making process in a car following scenario, in which each driver aims at keeping a fixed time-headway with respect to the preceding vehicle. When analyzing stability, the approximation of delays was done by using the asymptotic (limit) properties of distributed delay terms and the Taylor series expansion. The spectrum was computed by means of the QPmR toolbox. The authors inter alia found that the dynamics can exhibit the spectrum similar to NTDSs for some interconnection schemes, although the model does not fit in the standard NTDSs.

Single and double Hopf and the pitchfork bifurcation analyses were presented by Ding *et al.* [224] for an active control system of the ball valve in glue dosing processes for particleboard. For a double Hopf bifurcation, the multiple time scales method instead of the habitual Puiseux series was used, which is based on the following form of the solution of (1).

$$\mathbf{x}(t) = \sum_{i=1}^{\infty} \Delta^{\frac{2i-1}{2}} \mathbf{x}_i(T_0, T_1, \ldots) \tag{65}$$

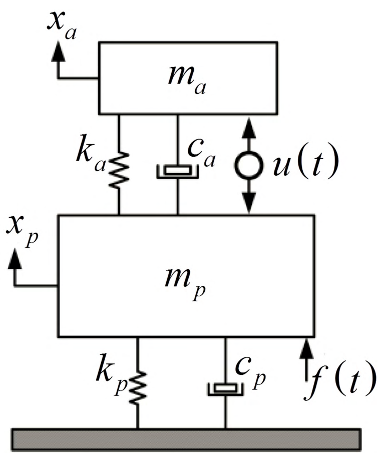

FIGURE 5. Delayed resonator (m_a, c_a, k_a) attached to the single degree-of-freedom system (m_p, c_p, k_p); displacement x_a is induced by the disturbance force $f(t)$ [229].

where $T_i = \Delta^i t$. The time derivatives of $\mathbf{x}(t)$ can be then calculated using the transformation below

$$\frac{\mathrm{d}}{\mathrm{d}t} = \sum_{j=0}^{\infty} \Delta^j \frac{\partial}{\partial T_j}. \quad (66)$$

For a particular delay deviation $\tau = \tau_c + \Delta\tau$ in the neighborhood of the critical point τ_c, the eventual formulas include arguments $T_j - \Delta^j(\tau_c + \Delta\tau)$ instead of $T_j, j \in \mathbb{N}$.

Takács and Stépán [225] studied the delay effect related to the existence of a traveling-wave-like motion of the tyre points in contact with the ground and relative to the wheel. They showed that the dynamics within the small-scale contact patch can have an essential effect on the global dynamics of a four-wheeled automobile on a large scale. Parameter-stability charts were determined by using the so-called D-subdivision method that identifies the Hopf-bifurcation points. Namely, $\Delta(s, \mathbf{p})|_{s=\mathrm{j}\omega}$ was separated into its real and imaginary parts, the zero points of which characterized the stability boundary curves parameterized by the angular frequency ω.

A Vandermonde/Birkhoff matrix methodology to study multiple imaginary poles [147]–[149] was applied to the control of a mechanical (vibration) system in [226].

The CTCR paradigm has been very popular when investigating particular DIS problems. For instance, Alikoç *et al.* [227] utilized the methodology of Ergenc *et al.* [228] to get the kernel curves when studying the train following problem with multiple communication time delays between the trains and the wayside control unit. The offspring curves and stability boundaries were further obtained by using the CTCR technique. The same combination of techniques was utilized by Eris and Ergenc [229] to get the complete resonance and stability maps for a two-delay delayed resonator with combination of speed and position feedback, i.e. $u(t) = q_1 x_a(t - \tau_1) + q_2 \dot{x}_a(t - \tau_2)$, see Fig. 5. Note that a couple of results have been recently obtained on the tuning and design of the delayed resonators for vibration suppression using feedback control laws, see e.g. Kučera *et al.* [230] and references therein; however, the problem goes beyond this survey. Olgac *et al.* [231] deployed the CTCR to study the ubiquitous blade/casing rub problem in turbomachinery and proposed the conditions to achieve stable rub interferences. The CTCR declared the complete stability outlook in the space of the operational and design parameters; hence, a combined DDS and parameter-dependent problem had to be solved. Hence, one can ascertain the exact stability interval for single-delay systems and the rigorous stability region for double-delay systems. The influence of the displacement-feedback delay with single delay and both displacement and velocity feedback delays on robotic actuator systems by using the CTCR methodology were discussed in [232]. The dominant pole distribution was computed as well. Zalluhoglu *et al.* [233] studied several alternative delayed feedback control schemes for stabilizing thermoacoustic instability in a Rijke tube. Although the paper was focused on the controller parameters tuning, the delay-dependent problem for the uncontrolled dynamics was also investigated. The novelty of this result is that the model was of a neutral type. The authors also validated their analytical findings on a laboratory-scale Rijke tube experiment.

A DDS analysis of a micro-grid system with a constant communication delay was presented by Gündüz *et al.* in [234]. Prior to the design of the stabilizing controller parameters based on the simple graphical parameter-plane method as e.g. in [203] and [225] (with a guaranteed spectral abscissa), the authors determined the stability delay margin based on gain and phase margins using the methodology presented in [72], [177], and [235]. Once the delay margin $L_{\max}$ and the corresponding crossing frequency $\omega_{L_{\max}}$ are found, the delay value satisfying the desired phase margin φ can be computed from (67).

$$L_{\varphi} = L_{\max} - \frac{\varphi}{\omega_{L_{\max}}} \quad (67)$$

The same techniques were applied to a single-area load frequency control system with a constant communication time delay in [236].

A combination of the Schur balanced truncation model reduction method, Lyapunov stability theory and LMIs was used to get delay stability margin (or a guaranteed spectral abscissa) for wide-area closed-loop interconnected power systems in [183]. Time-delay stability margins are investigated by Dong *et al.* [191] for wide-area measurement systems (as a part of interconnected bulk power systems) by means of the Jordan form, Taylor series expansion and the Schur simplification.

Biological sciences have not stood aside. Let us name just a few applications, besides the already introduced ones [104], [218], [223]. Breda *et al.* [92] dealt with physiologically structured populations of the Daphnia type. The authors revisited the pseudospectral approach [93] to compute the eigenvalues of the infinitesimal generator of linearized systems modeled by the Volterra functional equation and the FDEs, to study the local asymptotic stability of equilibria and relevant bifurcations. Delays in models were considered of lumped (i.e., discrete) and distributed types due to discontinuities in the vital rates at a maturation age and continuous age distribution, respectively. Beretta and Breda [237] further analyzed the occurrence of stability switches for population growth models with lumped and distributed delays. They inter alia found that for any choice of parameter values for which the lumped delay model exhibited stability switches, there existed a maximum delay variance beyond which no switch occurred for the continuous delay model. Moreover, the unstable delay region was as larger as lower the ratio between the juveniles and the adults mortality rates was. Diekmann *et al.* [238] presented a purely analytic study on the pole loci of the equilibrium of a cell population model governed by the characteristic quasipolynomial $\Delta(s) = s - d_{1,\tau} s \exp(-Ls) - d_0 - d_{0,\tau} \exp(-Ls)$ with $d_{1,\tau} \in (-1, 1)$. As a consequence, stability boundaries in the $(d_0, d_{0,\tau})$-plane were determined. Nakata [239] analyzed asymptotic stability of structured population dynamics models characterized by a scalar renewal integral (Volterra-type) equation and proved that if the negative feedback is characterized by a convex function, all characteristic roots lie in $\mathbb{C}_0^-$. In [240] and [241], Ünal *et al.* studied stability properties of a delayed Oregonator model governing the Belousov-Zhabotinskii reaction that represents a prototype for biochemical oscillators. More precisely, the authors found a region in parameter space that ensured the presence of sustained oscillations. Some necessary and sufficient conditions for the asymptotic behavior of the model were presented by using its unique positive equilibrium points. In their neighborhood, stability solutions were investigated, and Hopf bifurcation points and their crossing frequencies were discussed as well. Stability crossing curves in the delay space were computed via the DDE-BIFTOOL package [242]. Wirtinger-type double integral inequality was established to estimate the double integral term appearing in the derivative of the Lyapunov-Krasovskii functional with a triple integral term when analyzing stability of genetic regulatory network affected by time-varying delays in [243].

Note, however, that many results (especially from biomechanics) dealt with feedback stabilizing or control strategies; for instance, in order to get stable human postural balance [2], [3], [190], [244].

Selected results covered in this section, especially those not included in Table 1 or Table 2, are summarized in Table 3.

VIII. DISCUSSION AND OPEN PROBLEMS

The questions developed in the planning phase of this literature review study are concisely discussed in this section.

A. CURRENT STATE OF RESEARCH ON THE SPECTRUM ANALYSIS FOR LTI-TDS WITH CONSTANT DELAYS

After reviewing the selected articles, our first observation is that most of the researchers have focused on RTDSs with point-wise delays, see Tables 1 to 3. The authors of this survey believe that this is because of unfriendly and messy properties of NTDSs - the reader is referred to Propositions 2 and 3. Especially, a non-smooth behavior or even discontinuity of Σ_{ess} with respect to variations in delays constitutes an obstacle that many researchers found insurmountable. Similarly, systems with distributed delays have been widely neglected as well; however, such systems have many interesting and practically usable features, mainly in the mechanical engineering field, see e.g. [229], [230], and references therein. Nevertheless, there have been some articles dealing with these two families of systems and models; for instance, Bonnet *et al.* [51] and Nguyen *et al.* [117], [118] presented some nice semi-analytic results on the pole loci approaching the imaginary axis for NTDSs, Xu and Wang [106] used their technique based on the complex geometry to determine the dominant spectrum of a neutral type. Avanessoff *et al.* [120] provided researches and engineers with a software computing the complete spectrum of a NTDS. Zalluhoglu *et al.* [233] used the CTCR paradigm to solve the practically-oriented problem on a model evincing neutral delays. Lehotzky and Insperger [48] computed the rightmost part of the spectrum for distributed-delay systems. Breda *et al.* [92] and Beretta and Breda [237] utilized the pseudospectral approach to determine stability of population models. Kammer and Olgac [169] solved the DDS problem for dRTDSs via the CTCR, or it is worth highlighting the work of Michiels and Ünal [130] on FIR filters, to name just a few. Thus, both the aforementioned areas need more attention of researchers, due to a relatively limited volume of literature.

Another observation in this study is that many researchers have used some kind of approximation when computing the pole loci or determining the system stability based on the spectrum.

Discretization-based techniques (like PsC, SD, SE, SLT, NI, etc.), the Puiseux series expansion, Padé approximation, semi-analytic methods (see e.g. [161]) and many others simplify or approximate the original model in the first place. In fact, exact analytic techniques can be used only in special cases. For example, the use of the Lambert W function is only applicable for systems with commensurate delays and simultaneously triangularizable matrices. In addition, Möbius or Rekasius transformations are exact only for poles located on $\mathbb{C}^0$. And the parameter plane method for the stability boundaries determination can be practically applied only to systems with a low number of unknown parameters. As a matter of fact, it is extremely arduous to cope with the solution of the nonlinear transcendental eigenvalue problem (3) in general, and only very simple models were considered – e. g. [116], [141], [238] The disadvantage of some results can also be viewed in that only a partial solution was achieved; for instance, the delay margin instead of complete stability windows was determined, see e. g. [167], [177], [234].

Last but not least, we have observed that academic/engineering applications of the presented methods prevail in the field of mechanical engineering, followed by those in communication systems and biology or biomechanics.

Naturally, one wants to mostly know how the delay values influence the system stability using the knowledge of the purely imaginary poles and the corresponding delays.

B. OPEN PROBLEMS IN THIS FIELD

As it is clear from the issues discussed so far, there are a considerable number of open problems related to the spectral analysis of LTI-TDS. Let us name just a few.

First of all, the problem introduced as the primary one in the preceding subsection has to be attacked first. Although NTDSs or models with distributed delays have been studied by some authors, this review study does not include any research result on the spectrum of a dNTDS. Hence, it is desirable to consider this family of systems in a future research, regardless of a particular task. Especially, it would be useful for practitioners to equip software for spectrum (poles loci) computation [52], [120], [220], [242], with explicit tools specifically for distributed delay systems, or to extend some crucial results on TDS spectrum such as [105], [111], [116], [126], [165]. For NTDSs, it would be also attractive and challenging to investigate the sensitivity of pole loci lying exactly on $\mathbb{C}^0$ with respect to infinitesimal delay variations, mainly for cases with multiple roots [154].

Similarly, research results are lacking for higher order systems, or those with multiple or incommensurate delays. For instance, works of Nguyen *et al.* [117], [118], presenting useful explicit results for pole loci near the imaginary axis were focused on commensurate delays only. Much can be done with the Lambert W function as well. Cepeda-Gomez and Michiels [110] studied pole branches for a second order RTDS model only. Choudhary *et al.* [112] stated that there are many roots which correspond to the principal branch $k = 0$ and the $k = -1$ one, and that it is difficult to identify the rightmost ones among these several roots. Hence, the authors considered this task as a topic for further research. A natural question that can be raised is whether the Lambert W function can be used for higher-order, NTDSs or even incommensurate delay systems after some mathematical tricks. The same question applies to the Puiseux series expansion and other techniques for analyzing multiple roots on $\mathbb{C}^0$ (see e.g. [154], [159]), Explicit integral estimates of the fundamental function and its derivatives obtain only for ordinary differential equation with constant coefficients by means of the Cauchy W-method in [145] can be extended to delay differential equations with constant coefficients, as suggested by the authors of the method. Tweten *et al.* [91] noted that they were not aware of the paper extending the Legendre collocation method to arbitrary delays by implementing the same techniques used for the spectral method, and that future comparative studies for long, distributed, and arbitrary delays would be a nice follow-up to their paper. Breda concluded his paper [116] with the statement that the extension of analysis made in the paper towards several, apparently simpler, directions may have led to a further understanding eventually useful to tackle the general case; thus, the author planned to study equations with multiple delays. Promising findings from these upcoming studies can be looked forward to.

The fact that plenty of results are valid only for TDSs with commensurate delays (see Tables 1 and 2) can also be handled in another way. Namely, analytic or approximation techniques (especially, from the field of complex analysis) can be used to achieve a sufficiently accurate transition from incommensurate to commensurate delay models. Some attempts to cope with this task were already made in [189] and [190].

Another gap for researchers in the field is the transition of results solving the DIS problem to the task of DDS. The knowledge of Ω_c can be used to determine the crossing delays in some cases, as was done e.g. by Alikoç *et al.* [227] or Eris and Ergenc [229] by means of the Kronecker multiplication technique.

Some techniques suffer from mathematical complexity ([114], [137]) or rather high computational burden (e.g. due to sparse matrices of a high dimension, see the discussion in [84]); therefore, it would be desirable to use better hardware or software tools and to employ advanced programming skills for code optimization. For instance, distributed computations on graphical cards by means of the Compute Unified Device Architecture (CUDA) or the Open Computing Language (OpenCL) can be utilized. Milano [138] implemented the Shur method by the use of a GPU-based parallel computing and QR factorization was employed to speed up the computation. To achieve this, both high-performance computer and sophisticated programming skills were required.

It is quite surprising that the authors of this review have not found any result implementing some advanced optimization technique (such as genetic algorithm, particle swarm optimization, ant colony optimization, etc.) used to solve a spectral analysis task; despite the fact that some optimization problems can be defined. For example, the spectral abscissa can be found by the solution of problem (6) subjected to (3). There are other challenging tasks, see e.g., (31) for the condition of the SLS method, (45) for the searching of a stable model minimizing the H_2 norm, and the work of Otten and Mönnigmann [164] who proposed an optimization method for parametrically uncertain systems.

As introduced above, applications of the surveyed methods can be found in mechanical engineering, informatics, transportation and biology; however, no economical application has been found – despite of the fact that delayed economical models can be assembled [245]–[247]. From the global and everyday-life perspective, such a research would be highly useful.

Eventually, let us introduce some other specific research topics and specific problems raised by the cited authors. According to the best knowledge of Lehotzky and Insperger [48], detailed theoretical convergence analysis did not exist for the PsT and SE methods; however, precise theoretical convergence analyses were provided in [93] and [248] for the PsC and the SLT methods, respectively. Vyasarayani *et al.* [83] stated that it was worth investigating why the mixed Fourier basis performed worse in terms of the convergence compared with shifted Legendre and shifted Chebyshev basis. Ozoegwu [100] proposed to extend his line of research by generalizing vector numerical integration scheme for any order of approximation and apply the generalized result in solution of the milling problem with the aim of arriving at a generalized $\mathbf{T}_N$ of general order such that given any value, this matrix could be generated directly instead of passing through the rigorous derivation of the polynomial constant vector. When such a simplification is made available, it would be efficient to study the actual trend of accuracy beyond fourth order. The author also noted that there was a possibility to reduce the accuracy at higher order due to error akin to the Runge phenomenon; however, this did not rule out the importance of general-order and probably conclusive investigation of vector numerical integration schemes. Chen and Dai [105] stated that the estimation of a suitable number of poles inside the open disk and the efficient computation of the rightmost poles were remaining for their future work. Zalluhoglu *et al.* [233] claimed that their current research was focused on the transition of the gained knowledge to more elaborate combustors.

IX. CONCLUSION

The presented survey in the form of a literature overview has been focused on the analysis of linear time-invariant time-delay systems with constant delays related to their spectrum that provides one of the very basic and important information about the system stability and dynamics. Selected research results published mostly in the recent five years have been given to the reader with the objective to show the most updated information. The paper has been structured such that results dealing with pole loci have been followed by stability studies and, eventually, by academic and/or engineering applications. Open tasks, research gaps and some suggestions for the future research on related topics have also been concisely discussed. This study can be useful for researchers and practitioners in order to utilize the surveyed techniques and methodologies. Due to the complexity and comprehensiveness of the considered subject, it has been almost impossible for the authors to cover all the results that could be found in the literature. In the future, the authors intend to follow up on this study with a survey on eigenvalue-oriented control synthesis methods for the topic of interest.

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